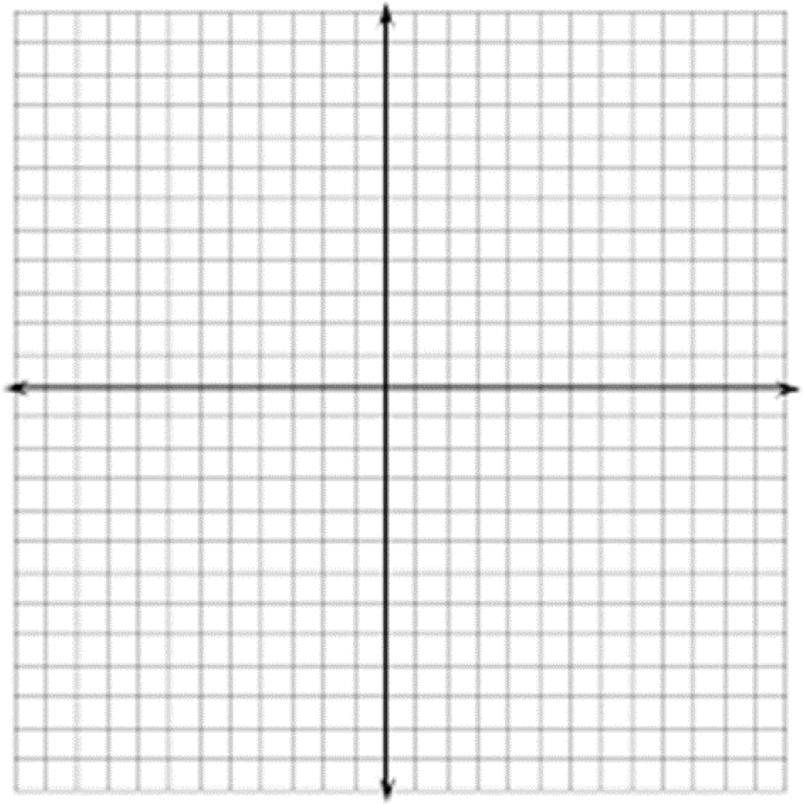
Quadratic Functions

Bryan Stanton – Charlotte Country Day

Exploration

- Insert each of the inputs in the left-hand column of the table into the function $f(x) = x^2$ to obtain an output. Then plot each $(x, f(x))$ point on the coordinate plane to the right and connect the points.

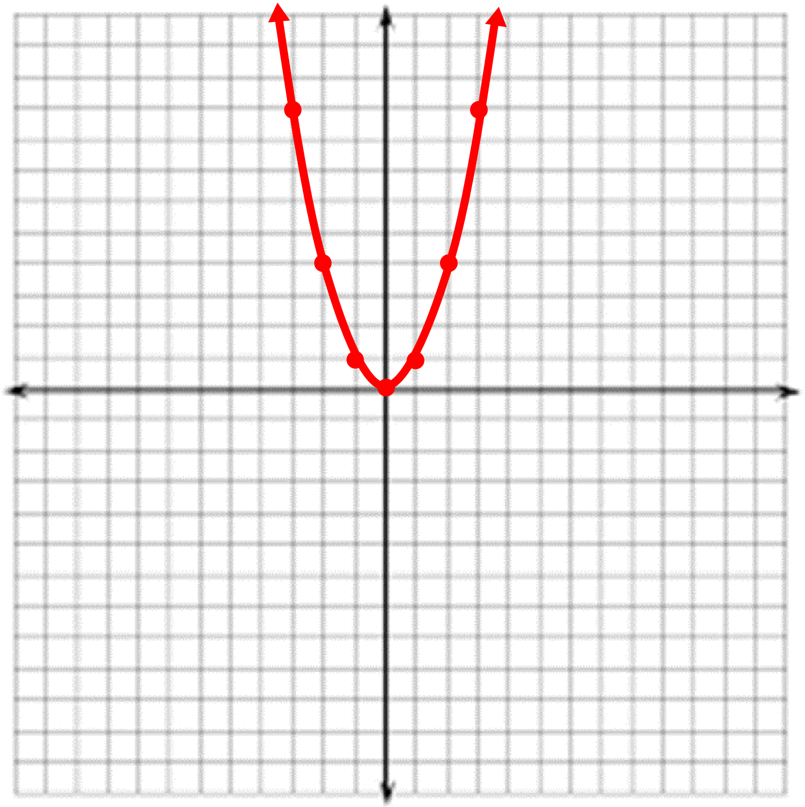
x	$f(x)$
-3	
-2	
-1	
0	
1	
2	
3	



Quadratic Functions

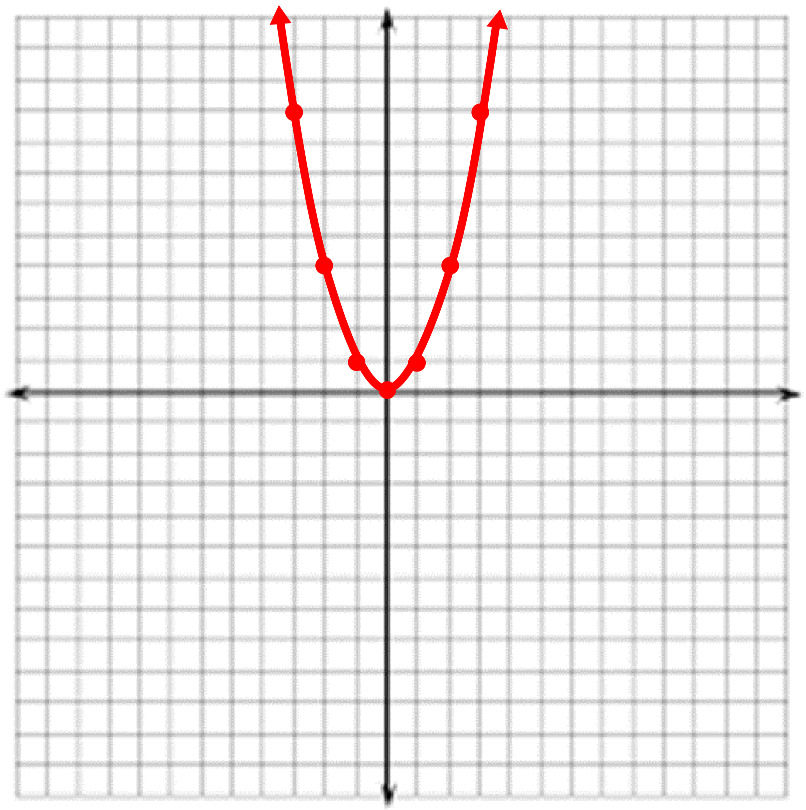
- The function $f(x) = x^2$ is a known as a **QUADRATIC FUNCTION**. The shape of the graph of a quadratic function is known as a **PARABOLA**.

x	$f(x)$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



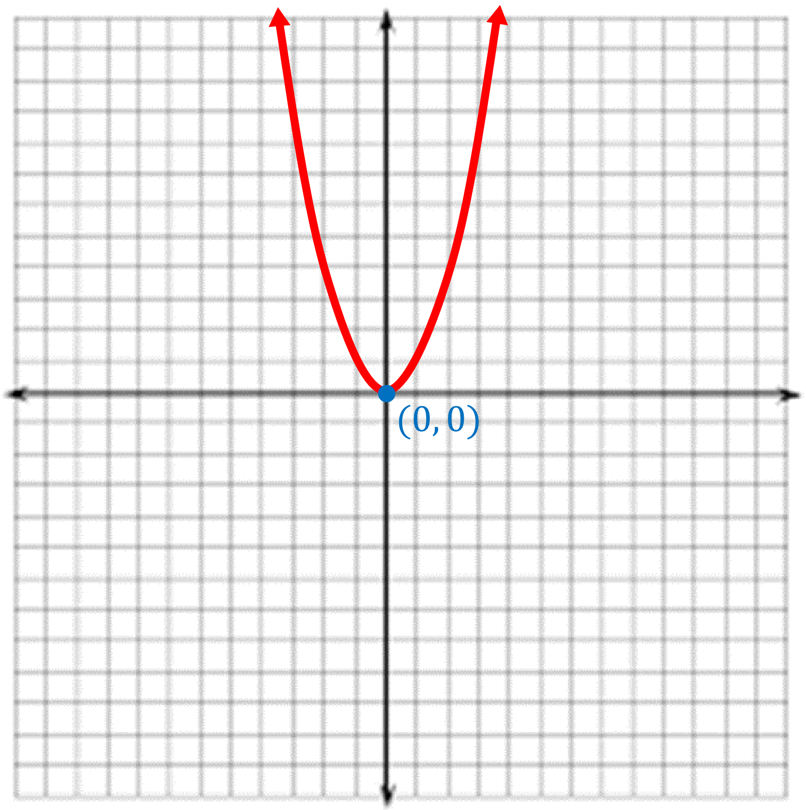
Important Note

- Any function in which x is being raised to an even exponent (ie $f(x) = x^4$, $f(x) = x^6$, etc.) will have the same basic shape as the graph of $f(x) = x^2$.



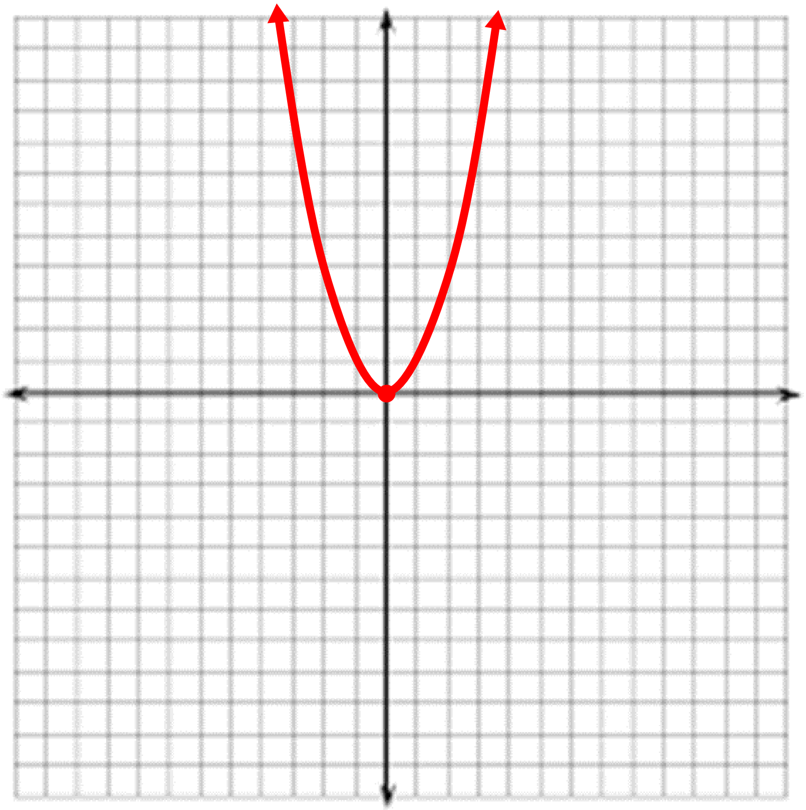
Quadratic Functions

- The point where the parabola changes directions is known as the **VERTEX** of the parabola. For the function $f(x) = x^2$, the vertex is located at the origin.



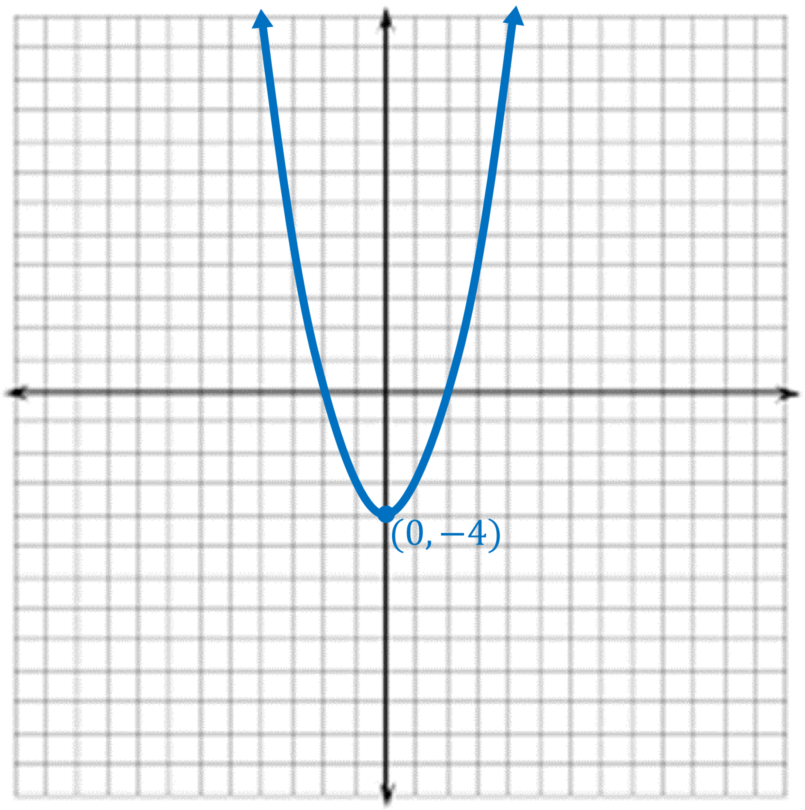
Quadratic Functions

- Now, $f(x) = x^2$ is a parent function like the ones we studies last unit and therefore we can transform its graph just like we did with other parent functions. If there was a similar function $g(x)$ in which $g(x) = x^2 - 4$, how do you think its graph differs from that of $f(x)$?



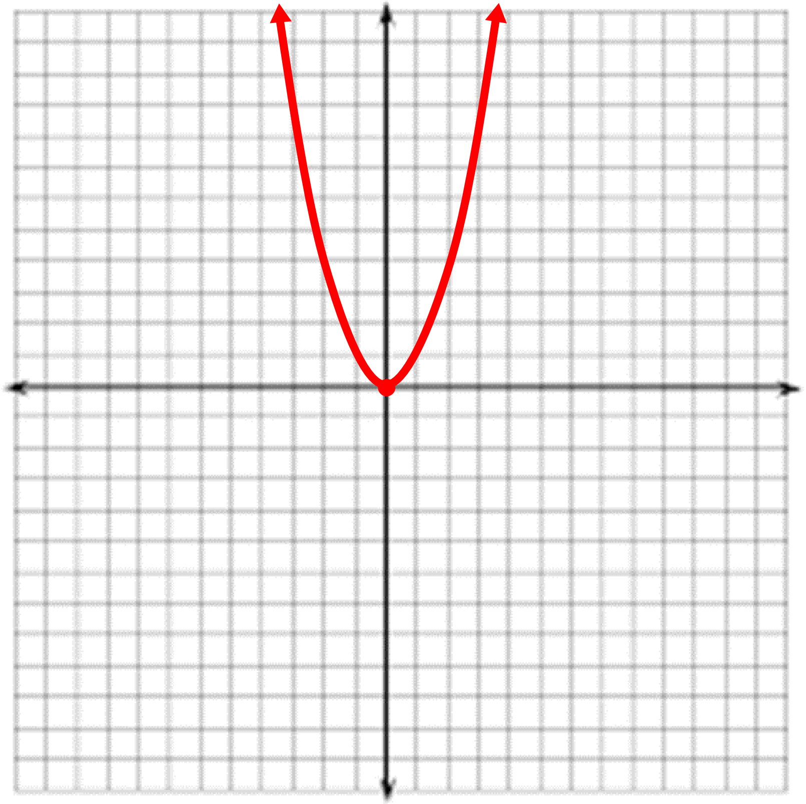
Quadratic Functions

- The "-4" of $g(x) = x^2 - 4$ would shift every point on the graph of $f(x)$ **DOWN** four units.



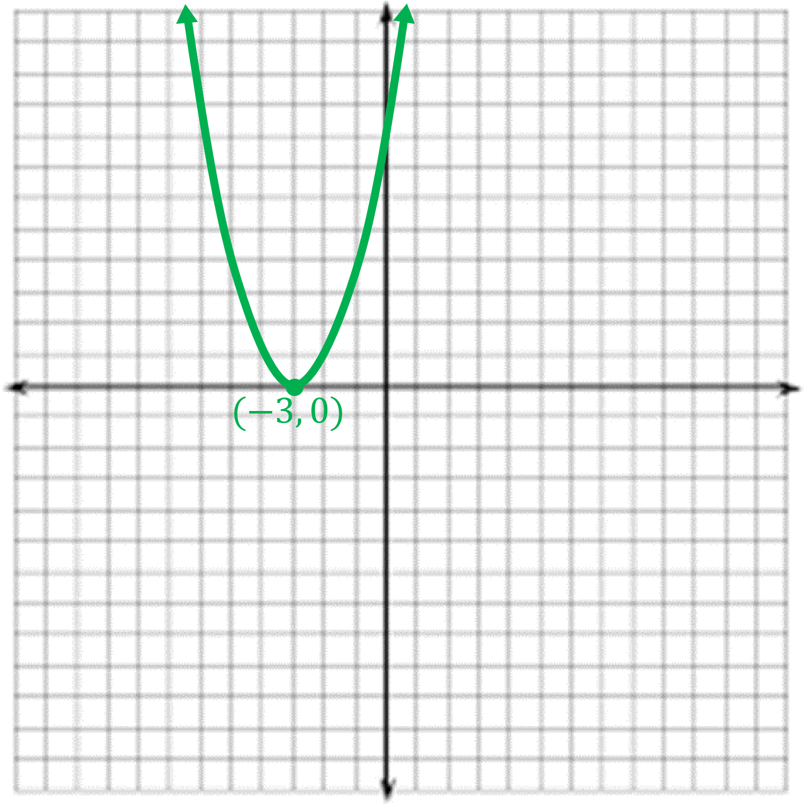
Quadratic Functions

- And if there was a similar function $h(x)$ in which $h(x) = (x + 3)^2$, how do you think its graph differs from that of $f(x)$?



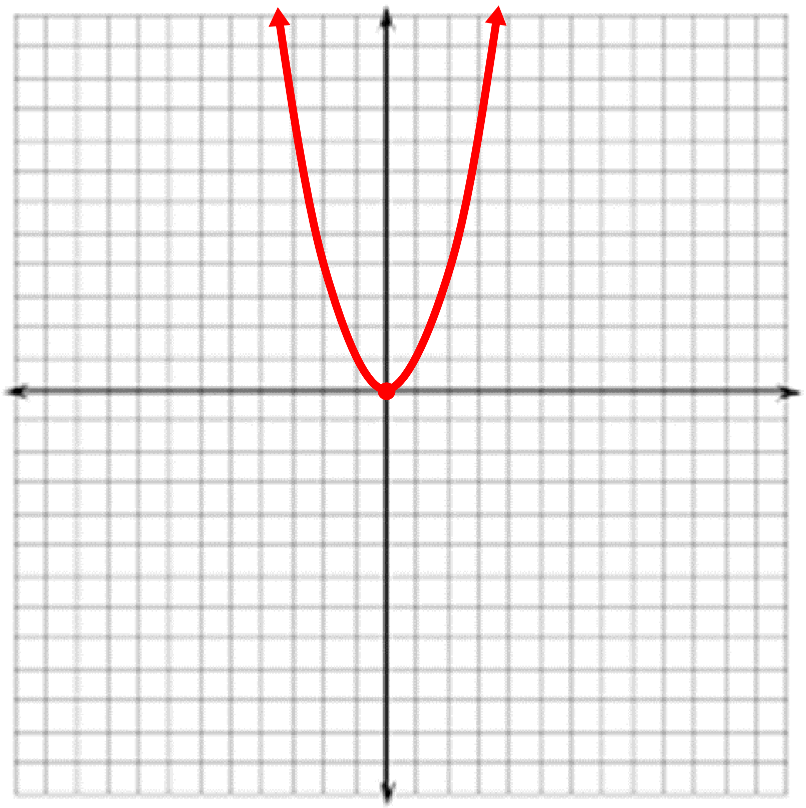
Quadratic Functions

- The "$+3$" of $h(x) = (x+3)^2$ would shift every point on the graph of $f(x)$ **LEFT** three units.



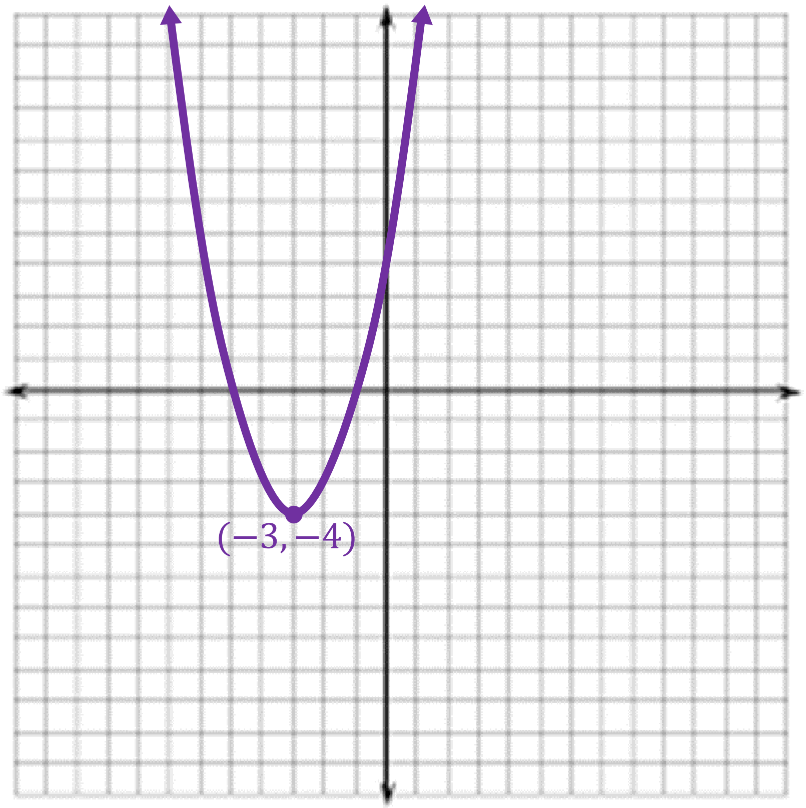
Quadratic Functions

- And if there was a similar function $p(x)$ in which $p(x) = (x+3)^2 - 4$, how do you think its graph differs from that of $f(x)$?



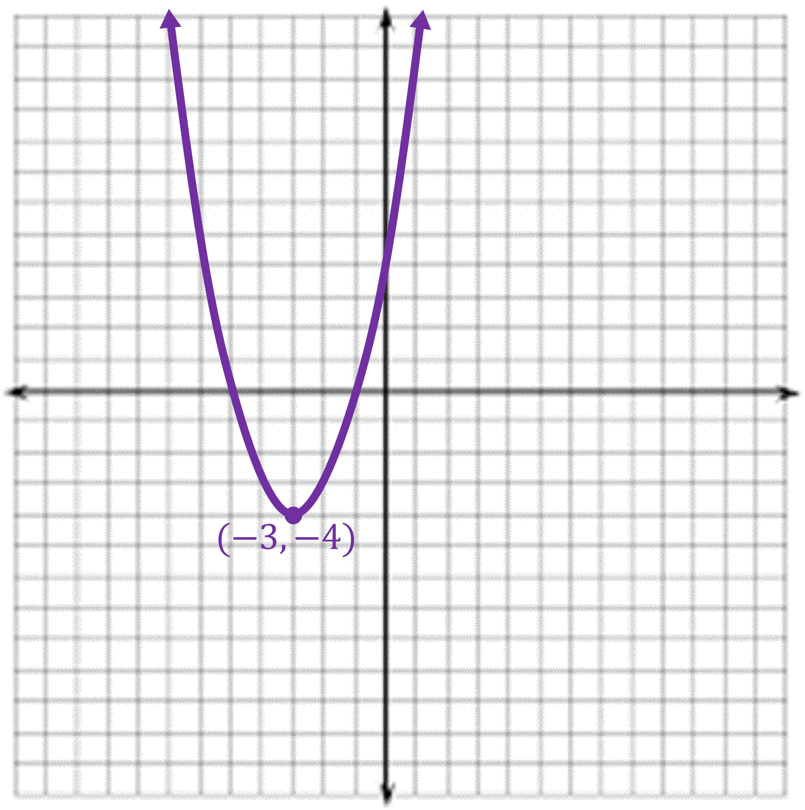
Quadratic Functions

- The "$+3$" of $p(x) = (x + 3)^2 - 4$ would shift every point on the graph of $f(x)$ **LEFT** three units and the "-4" would then shift every point on that graph **DOWN** four units.



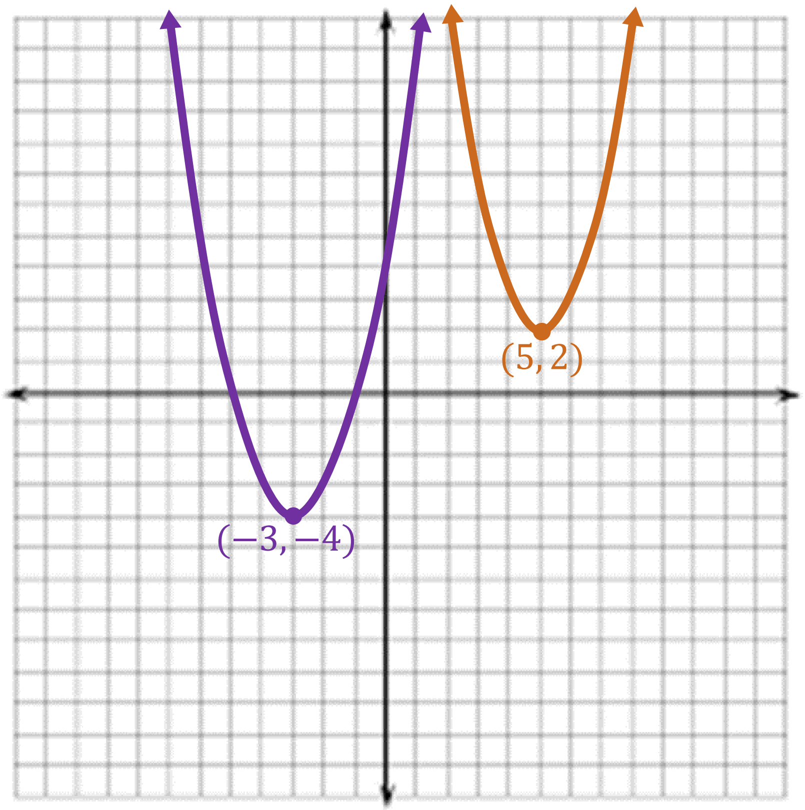
Quadratic Functions

- Notice where the vertex of the quadratic function $p(x) = (x+3)^2 - 4$ lies. Based on this observation, where do you think the vertex of the quadratic function $z(x) = (x-5)^2 + 2$ lies?



Quadratic Functions

- The vertex of the quadratic function $z(x) = (x - 5)^2 + 2$ lies at the point $(5, 2)$.



Quadratic Functions

- Now let's abstract this concept for the quadratic function below. Where would the vertex of this function be?

$$f(x) = (x - h)^2 + k$$

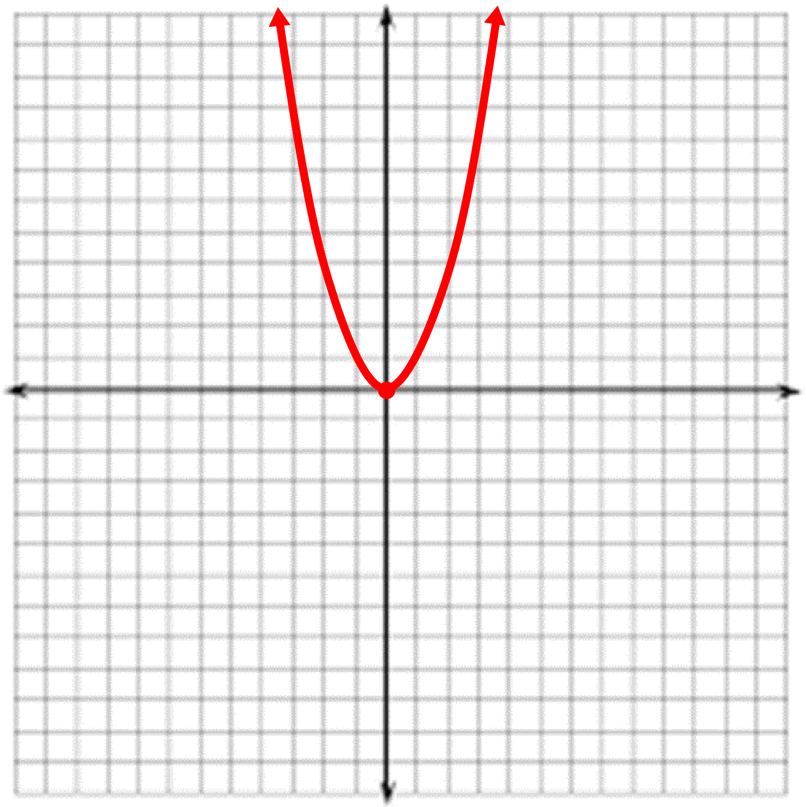
Quadratic Functions

- The vertex of any quadratic function in the form $f(x) = (x - h)^2 + k$ lies at the point (h, k).

$$f(x) = (x - h)^2 + k \qquad \rightarrow \qquad \text{Vertex: } (h, k)$$

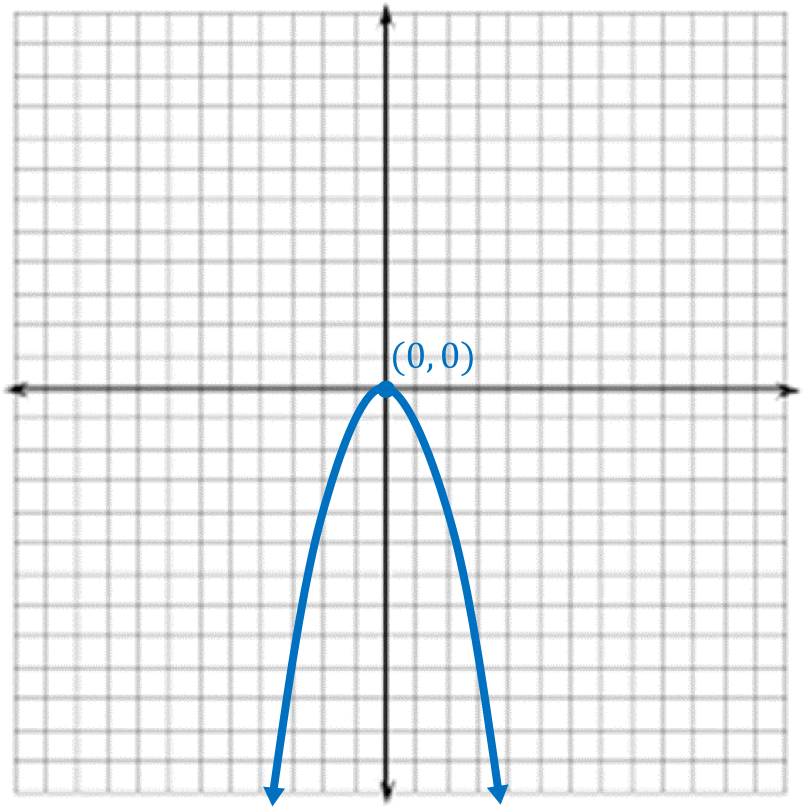
Quadratic Functions

- Now, if there was a similar function $w(x)$ in which $w(x) = -x^2$, how do you think its graph differs from that of $f(x)$?



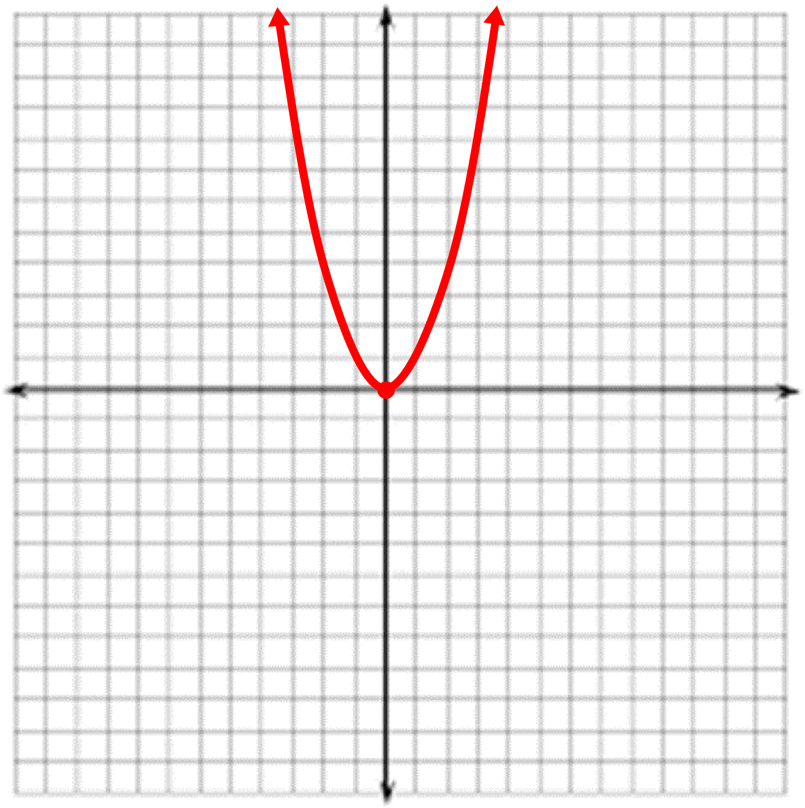
Quadratic Functions

- The "–" of $w(x) = -x^2$ would **FLIP** every point on the graph of $f(x)$ over the x-axis.



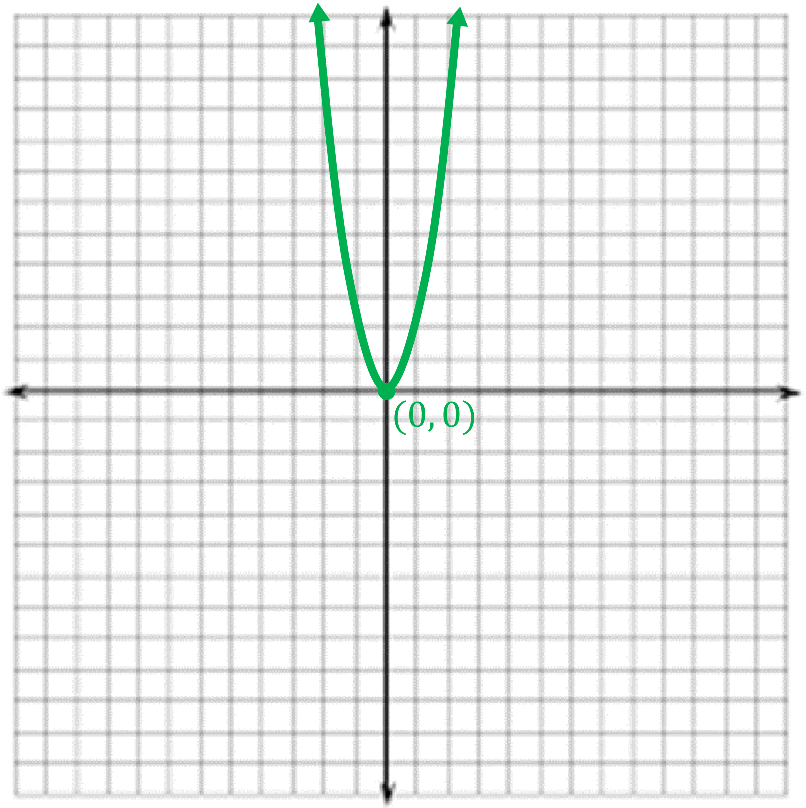
Quadratic Functions

- And if there was a similar function $v(x)$ in which $v(x) = 2x^2$, how do you think its graph differs from that of $f(x)$?



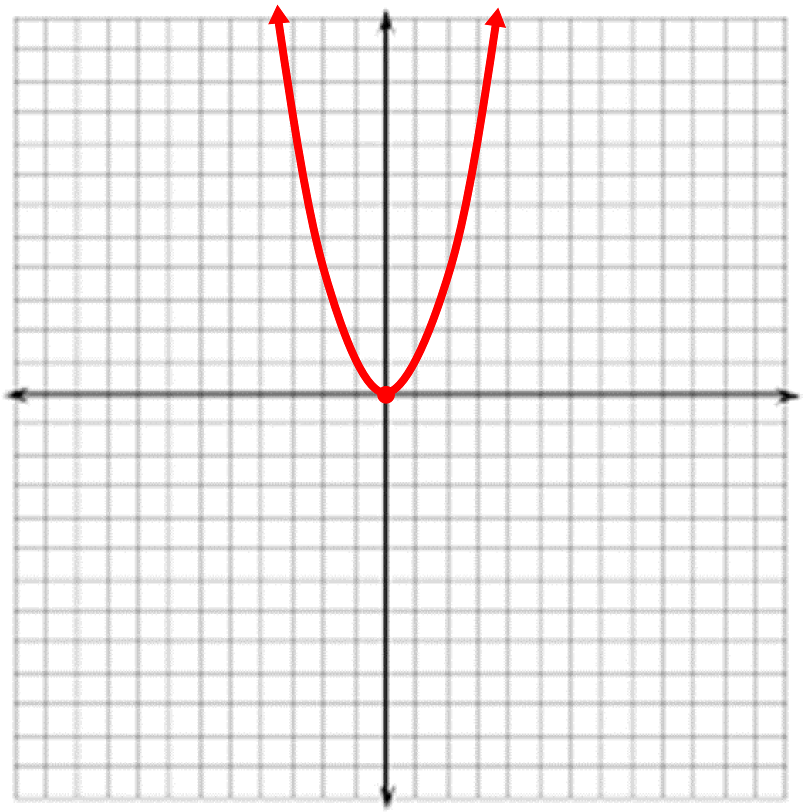
Quadratic Functions

- The "2" of $v(x) = 2x^2$ would **STRETCH** every point on graph of $f(x)$ twice as far from the x-axis.



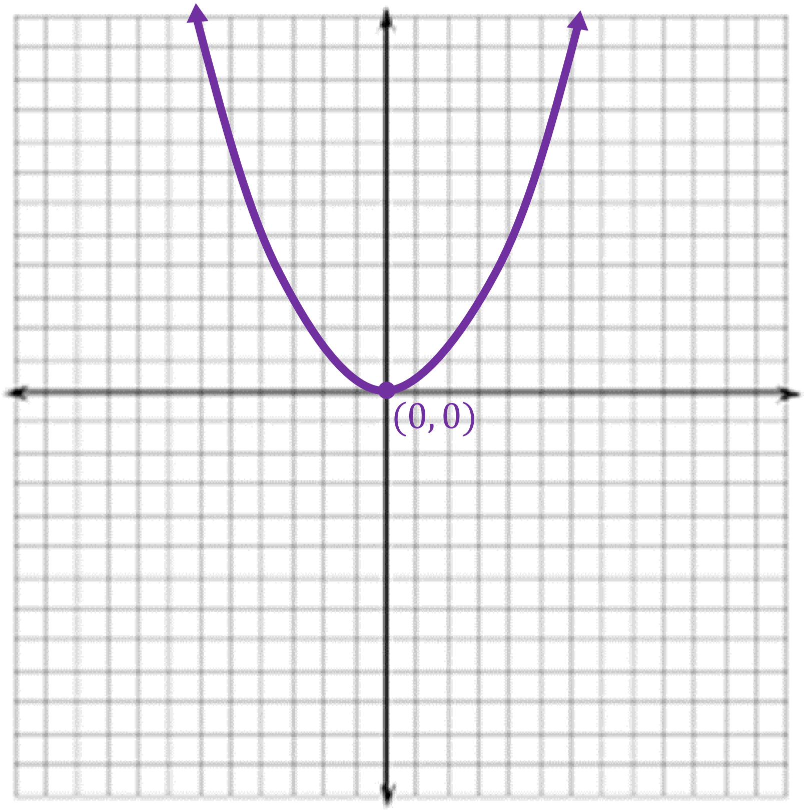
Quadratic Functions

- And if there was a similar function $r(x)$ in which $r(x) = \frac{1}{3}x^2$, how do you think its graph differs from that of $f(x)$?



Quadratic Functions

- The "$\frac{1}{3}$" of $r(x) = \frac{1}{3}x^2$ would **COMPRESS** every point on graph of $f(x)$ three times closer to the x-axis.



Quadratic Functions

- So if the x^2 term of a quadratic function has a coefficient, the coefficient can stretch the graph vertically, compress the graph vertically, and/or flip the parabola over the x-axis.

$$f(x) = ax^2$$

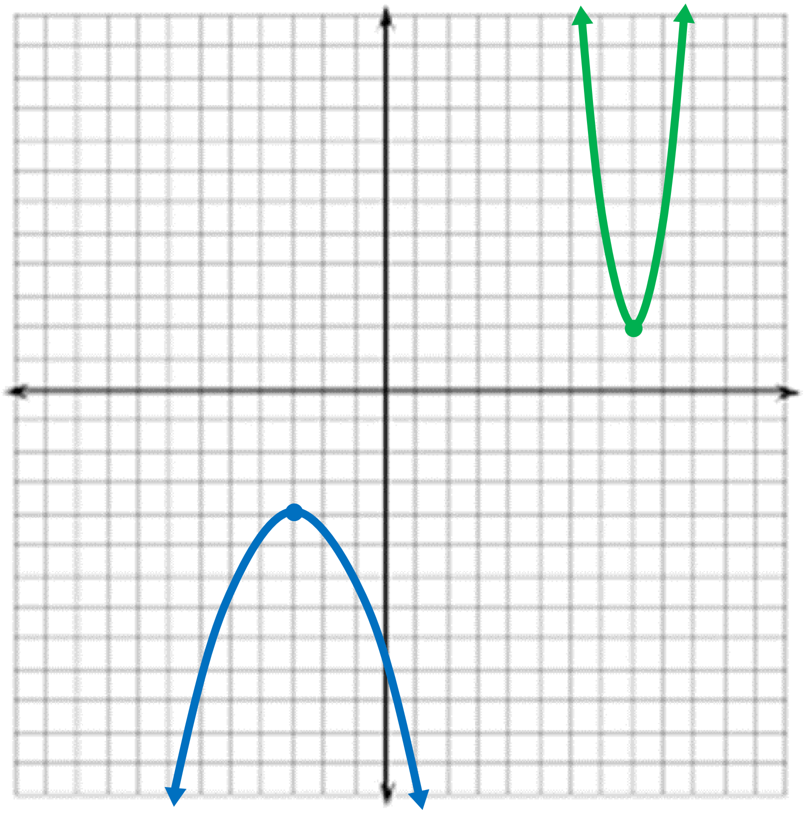
Quadratic Functions

- And combining the a-value with the h- and k-values from earlier, we get what is known as the **VERTEX FORM** of a quadratic function. It is named this way because we can easily pick out the vertex ((h, k)) from this form of a quadratic function.

$$f(x) = a(x - h)^2 + k$$

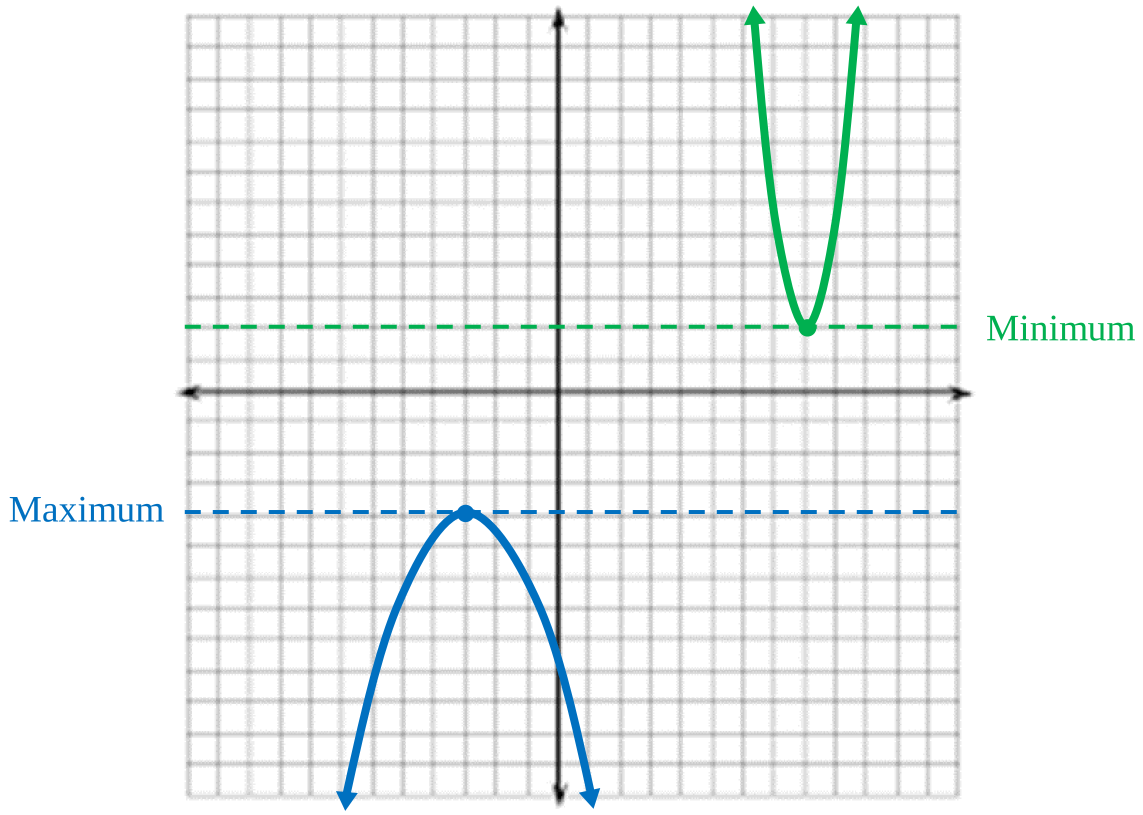
Important Note

- When the a-value is positive, the parabola will open upwards. The words that express a parabola opening upwards are "**CONCAVE UP**." When the a-value is negative, the parabola will open downwards. The words that express a parabola opening downwards are "**CONCAVE DOWN**."



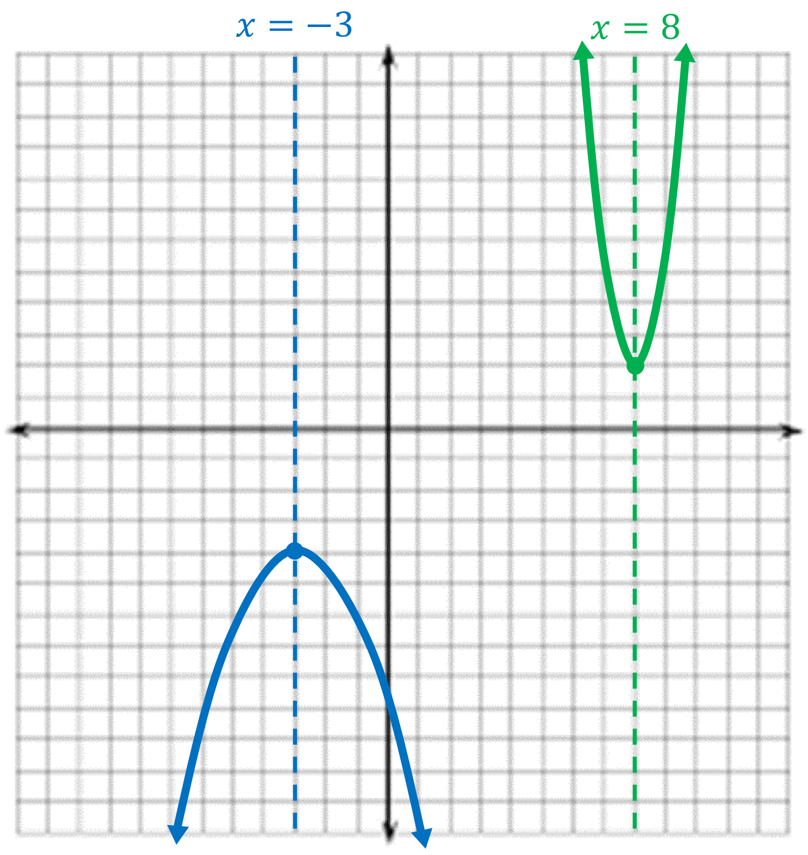
Important Note

- If the parabola is **CONCAVE UP**, the *y*-coordinate of the vertex will be referred to as the **MINIMUM** value of the quadratic function. If the parabola is **CONCAVE DOWN**, the the *y*-coordinate of the vertex will be referred to as the **MAXIMUM** value of the quadratic function.



Axis of Symmetry

- Every parabola is symmetrical about an imaginary vertical line known as the **AXIS OF SYMMETRY**. While the y-coordinate of the vertex is the maximum or minimum value of the function, the x-coordinate of the vertex is the location of the axis of symmetry.



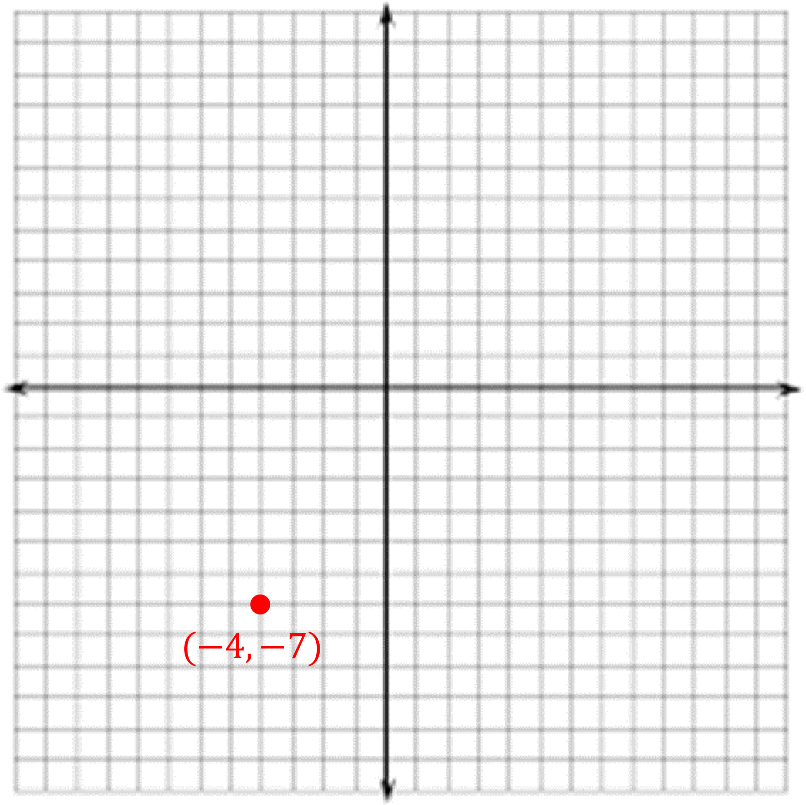
Graphing Parabolas from Vertex Form

- So let's say we wanted to graph the quadratic function below. How might we start this process?

$$f(x) = 2(x+4)^2 - 7$$

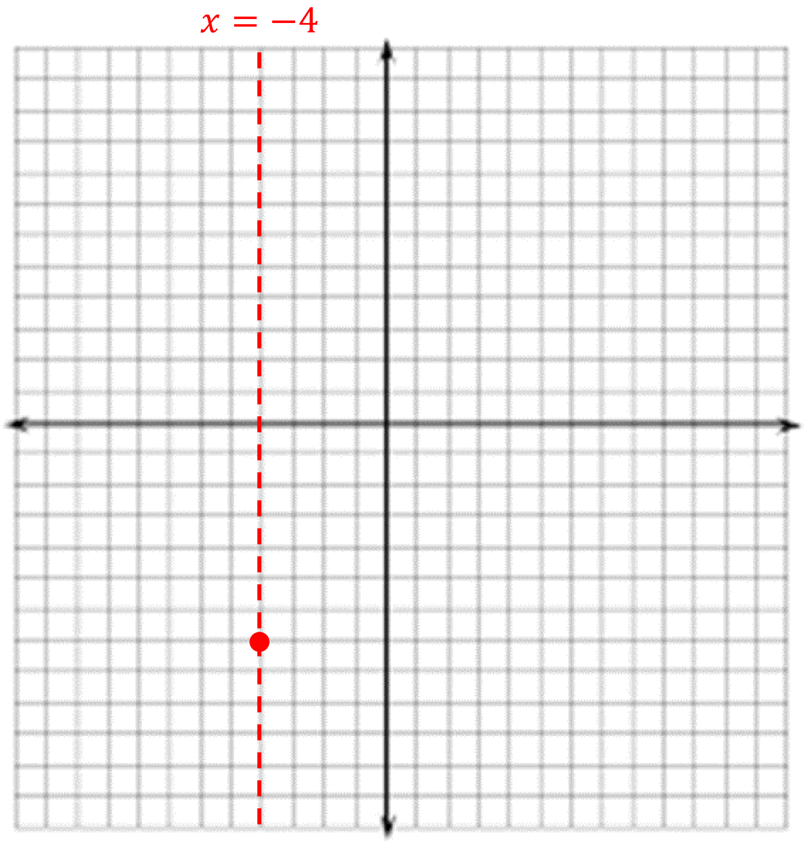
Graphing Parabolas from Vertex Form

- First, we could identify and plot the vertex of the function $f(x) = 2(x+4)^2 - 7$.



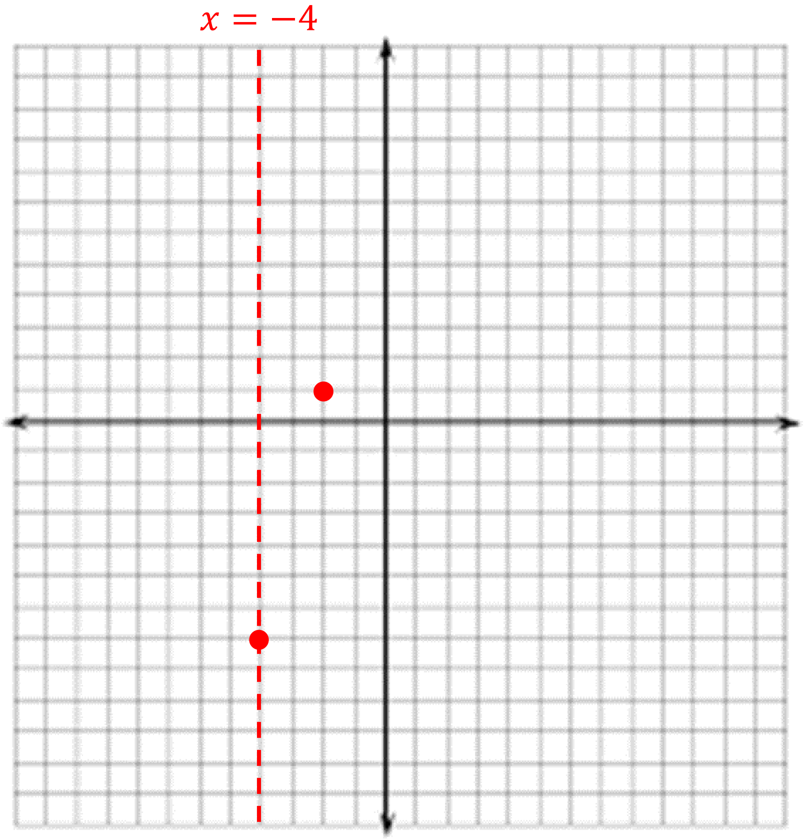
Graphing Parabolas from Vertex Form

- Next, we could draw a dashed axis of symmetry that passes through the vertex.



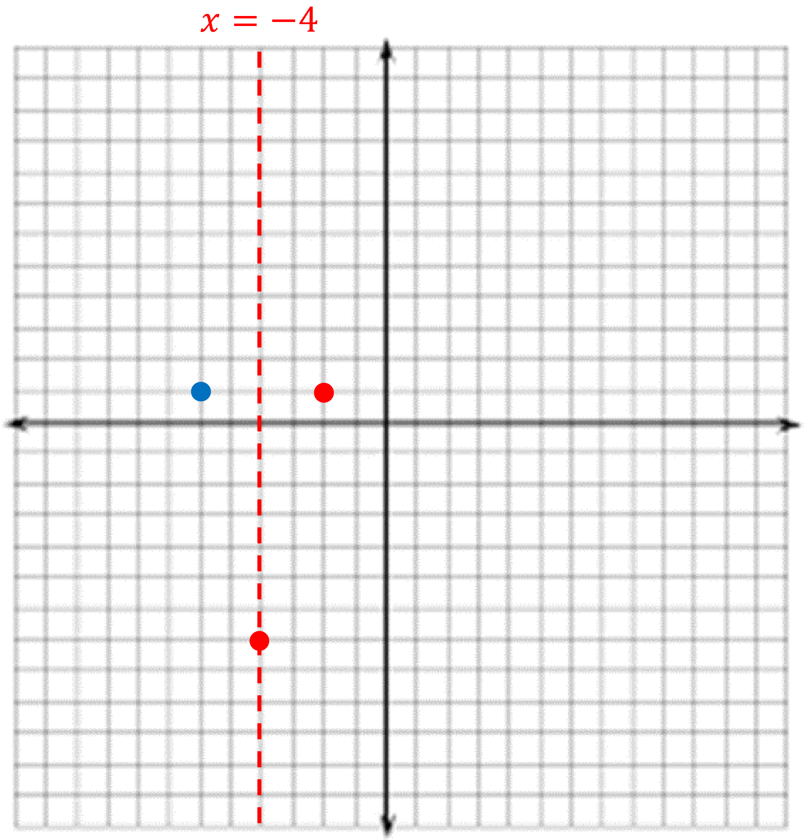
Graphing Parabolas from Vertex Form

- Next, we could pick an x-value to plug into function $f(x) = 2(x + 4)^2 - 7$ that is relatively close to the axis of symmetry. For instance, if we plugged in $x = -2$, we'd get function $f(-2) = 1$ and so we'd plot the point $(-2, 1)$.



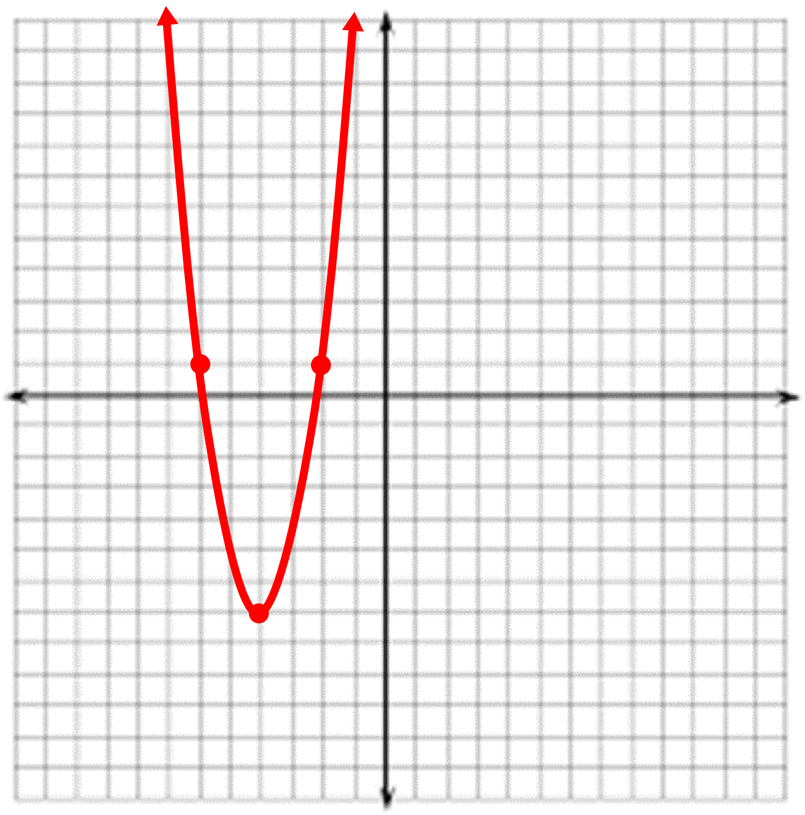
Graphing Parabolas from Vertex Form

- Now, since parabolas are symmetrical about the axis of symmetry, we could plot another point on the other side of the axis of symmetry that is the same distance away from the axis of symmetry as the point that we just plotted.



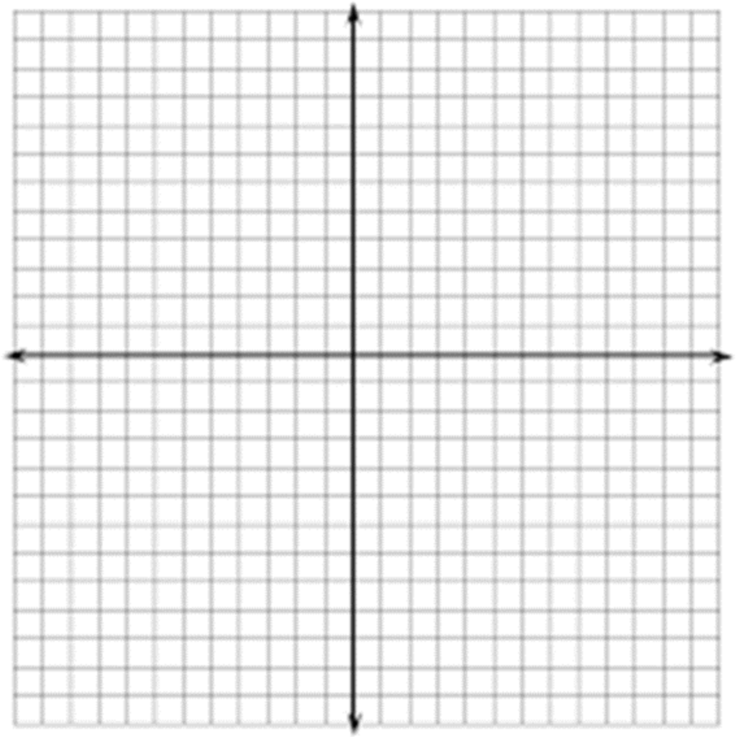
Graphing Parabolas from Vertex Form

- And last, we can connect the points we've plotted and extend the parabola in both directions.



Practice

- Sketch a graph of the quadratic function $g(x) = 3(x-5)^2 - 8$ on the coordinate plane below. Then state the domain and range of $g(x)$.



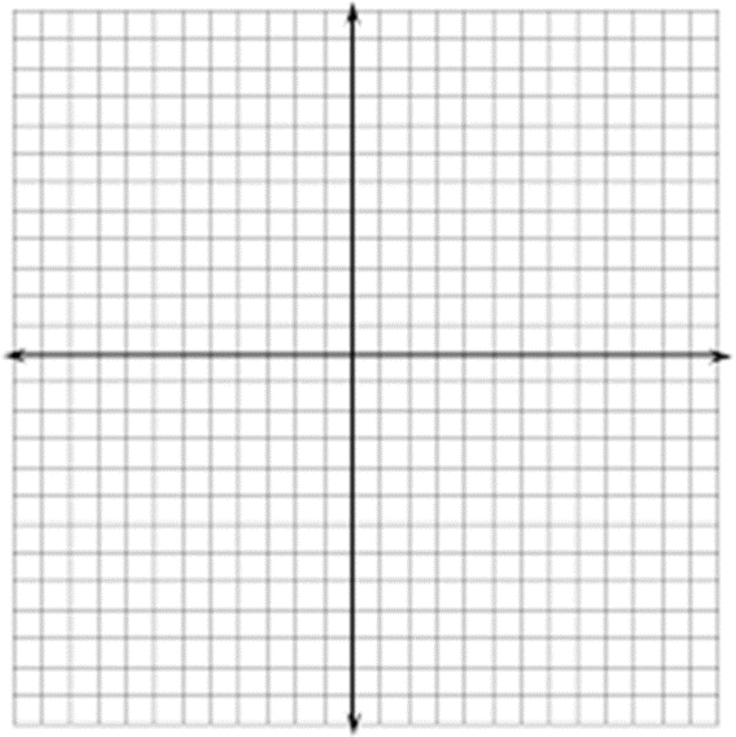
Important Note

- When you're choosing a number to plug into a quadratic function where the a-value is a fraction, it may help to choose a number that will make the $(x - h)^2$ part of the function divisible by the denominator of the a-value. For instance, choosing $x = 10$ would give us 6^2, which is divisible by 6.

$$g(x) = \frac{1}{6}(x - 4)^2 - 9$$

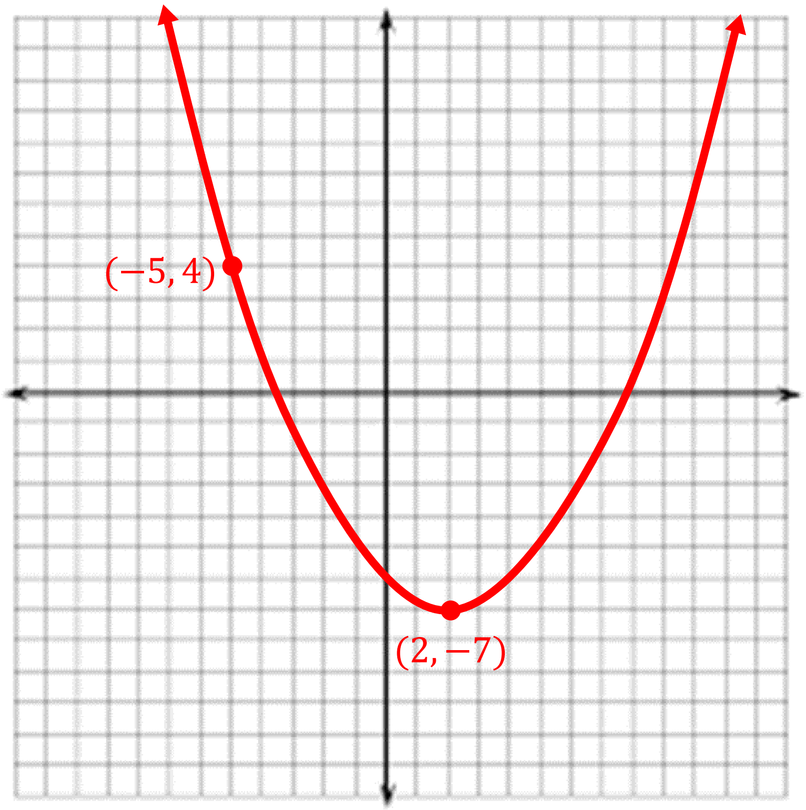
Practice

- Sketch a graph of the quadratic function $h(x) = -\frac{5}{4}(x+1)^2 + 3$ on the coordinate plane below. Then state the domain and range of $h(x)$.



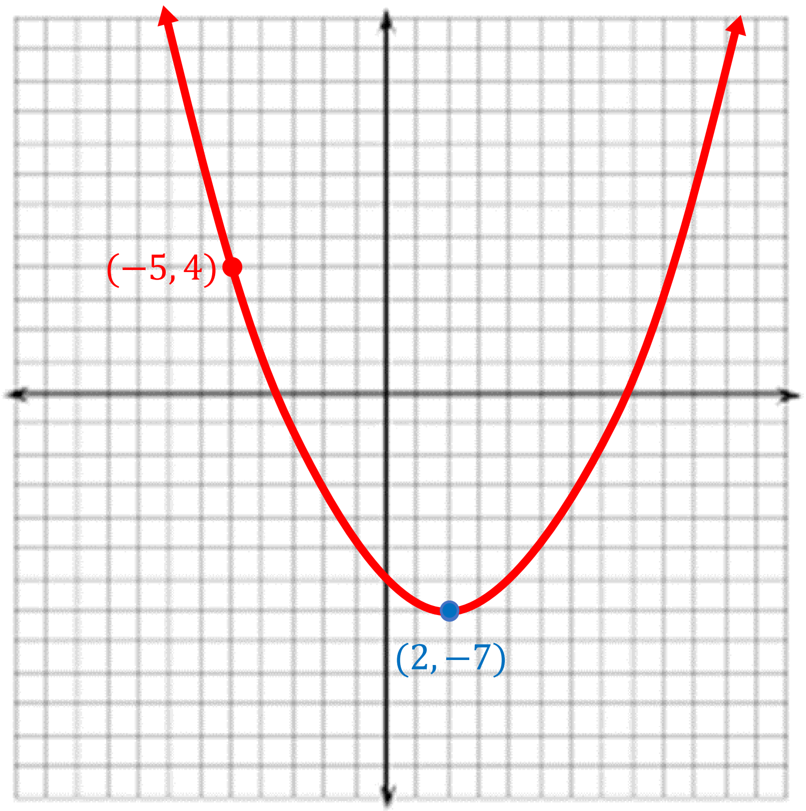
Quadratic Functions in Vertex Form

- While sometimes you'll be asked to graph a quadratic function when it's written in vertex form, other times you'll be given the graph of a parabola and you'll be asked to find an equation that represents the function. Let's try this with the graph of the quadratic function $f(x)$ below.



Quadratic Functions in Vertex Form

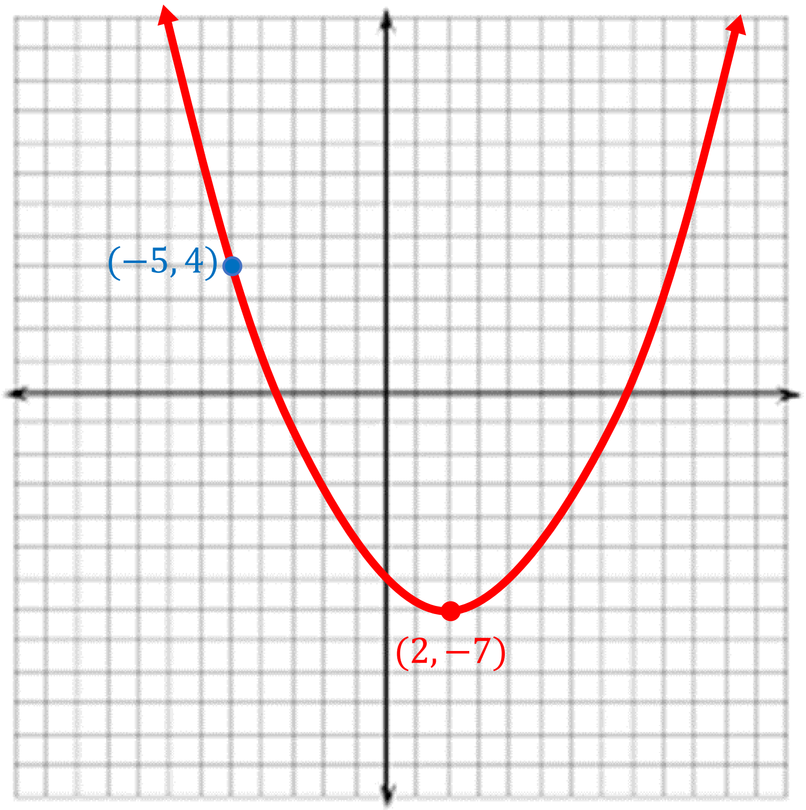
- First, since we know the vertex is at $(2, -7)$, we can say $f(x) = a(x-2)^2 - 7$.



Quadratic Functions in Vertex Form

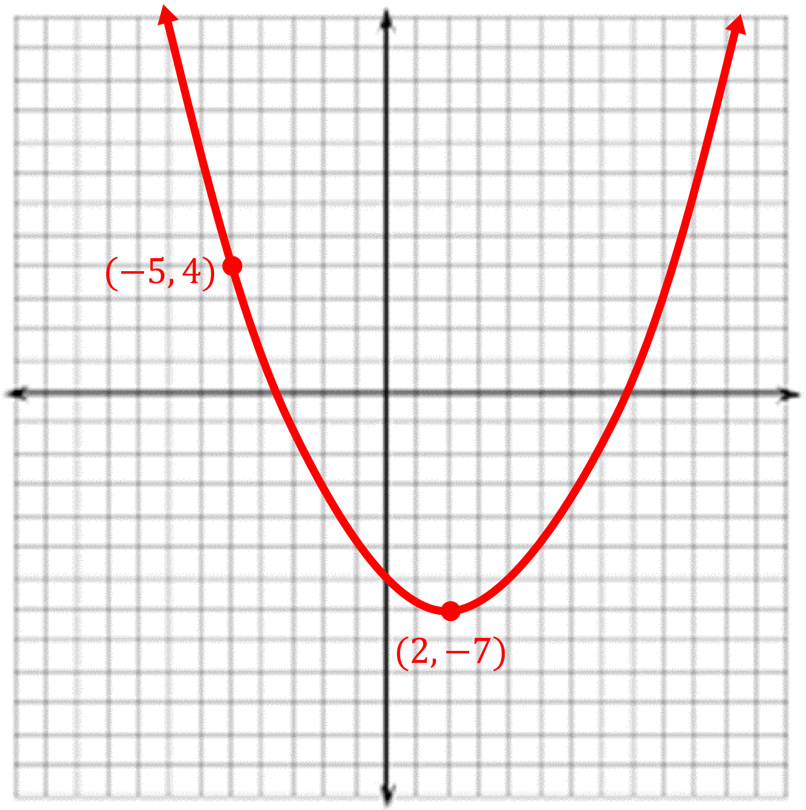
- Next, we can substitute **ANY** point that is not the vertex in for x and $f(x)$ to figure out what a equals.

$$f(x) = a(x-2)^2 - 7 \quad \rightarrow \quad 4 = a(-5-2)^2 - 7 \quad \rightarrow \quad a = \frac{11}{49}$$



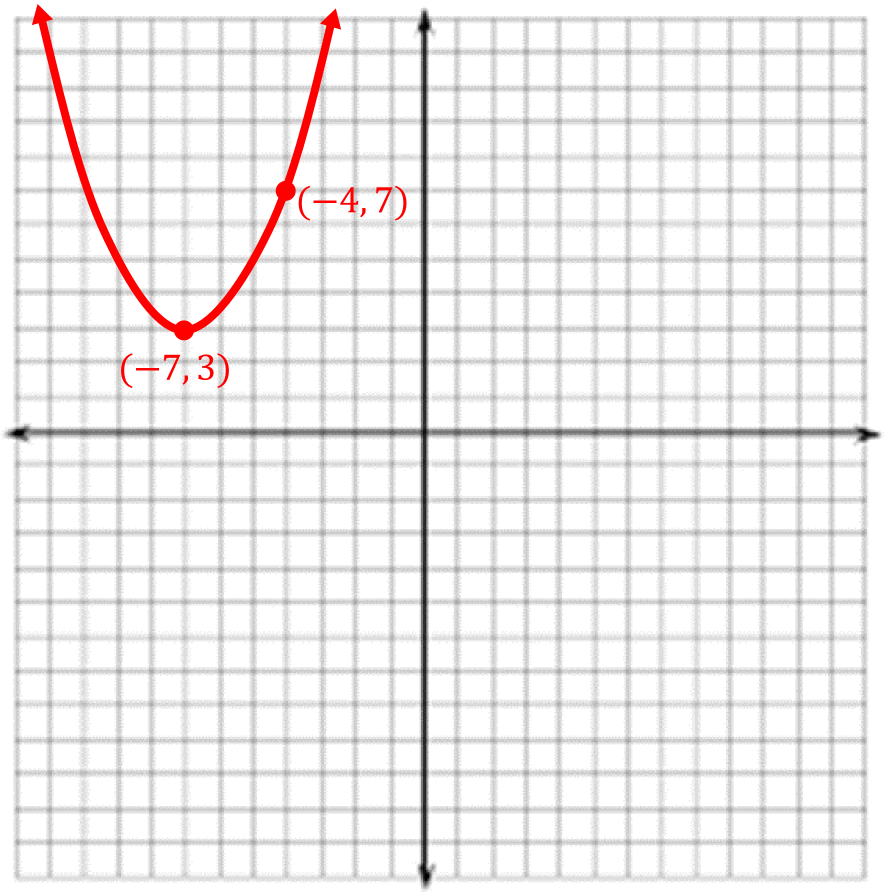
Quadratic Functions in Vertex Form

- And once we have the a-value, we can substitute it back in to get $f(x) = \frac{11}{49}(x-2)^2 - 7$.



Practice

- The graph below is of the quadratic function $f(x)$. Find an equation for $f(x)$.



Practice

- The quadratic function $p(x)$ has a vertex at the point $(2, 11)$ and passes through the point $(4, -6)$. Find an equation for $p(x)$.

Forms of Quadratic Functions

- The **VERTEX FORM** of a quadratic function is not the only form of a quadratic function that there is. There are two other equivalent forms that may be useful to us if we don't know the coordinates of the parabola's vertex. One of these forms is known as **STANDARD FORM** and the other is known as **FACTORED FORM**. The a-value for all three forms of a quadratic function will be the same.

$$f(x) = a(x-h)^2 + k \quad \leftrightarrow \quad f(x) = ax^2 + bx + c \quad \leftrightarrow \quad f(x) = a(x-r)(x-s)$$

Forms of Quadratic Functions

- Now, how do you think we could take a quadratic function that is written in vertex form and change it so that it is written in standard form?

$$f(x) = 2(x + 1)^2 - 5$$

Forms of Quadratic Functions

- First, we could square the binomial.

$$f(x) = 2(x+1)^2 - 5 \quad \rightarrow \quad f(x) = 2\left(x^2 + 2x + 1\right) - 5$$

Forms of Quadratic Functions

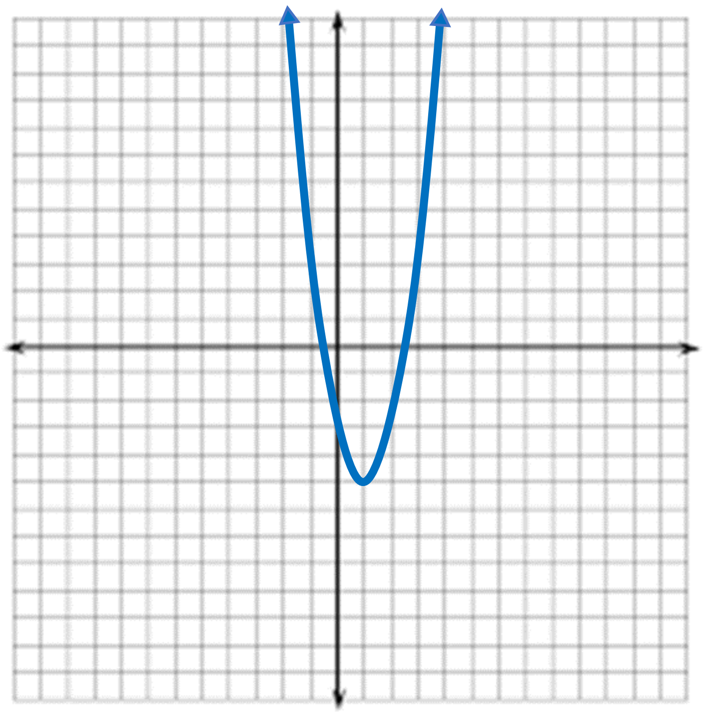
- And second we could distribute the a-value to all the terms inside the parentheses and then combine like terms.

$$f(x) = 2\left(x^2 + 2x + 1\right) - 5 \quad \rightarrow \quad f(x) = 2x^2 + 4x - 3$$

Standard Form of Quadratic Functions

- While a quadratic function written in vertex form helps us identify the vertex of the parabola, what do you think a quadratic function written in standard form helps us identify about the parabola?

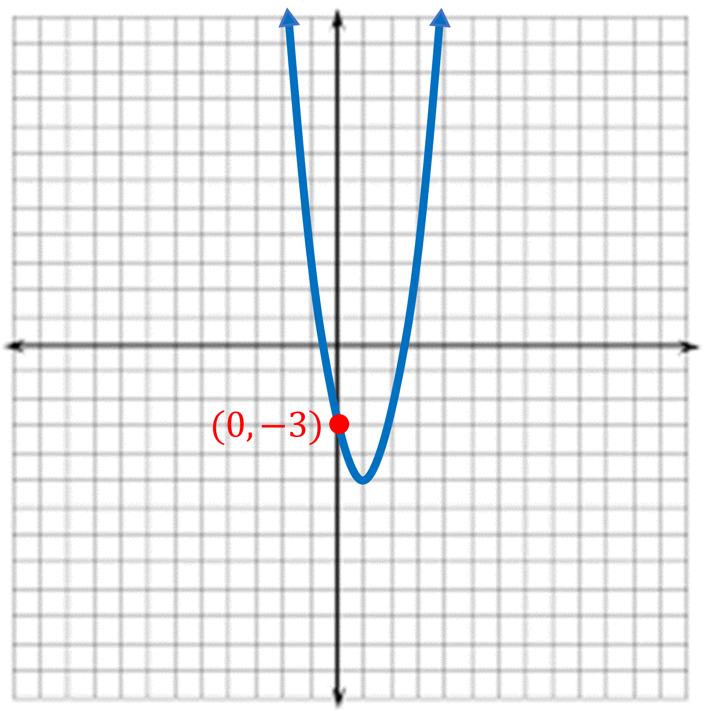
$$f(x) = 2x^2 + 4x - 3$$



Standard Form of Quadratic Functions

- The constant (which is the c-value) of a quadratic function written in standard form is the parabola's **Y-INTERCEPT**.

$$f(x) = 2x^2 + 4x - 3$$



Practice

- Convert the following function to standard form and then write down the function's y-intercept.

$$f(x) = -3(x - 5)^2 + 4$$

Exploration

- Now, let's say we knew the graph of a quadratic function ($f(x)$) passed through the points $(-6, -10)$, $(0, -4)$, and $(2, 6)$ and that none of these points was the vertex of the function. How might we find an equation for $f(x)$?

Exploration

- Well, since we don't know the vertex of the function, it doesn't make sense to start with the vertex form of a quadratic function. So let's start with standard form ($f(x) = ax^2 + bx + c$). Based on the points we are given ($(-6, -10)$, $(0, -4)$, and $(2, 6)$), what do we know about the graph of the function?

Exploration

- Since the y-intercept of the function is at $(0, -4)$, we know that $c = -4$. Now, what can we do to start finding the values of a and b?

$$f(x) = ax^2 + bx - 4$$

Exploration

- We could plug in one of the other points for x and $f(x)$ to get one equation and the other point in for x and $f(x)$ to get another equation. When we do this, we'll get a **SYSTEM OF EQUATIONS**.

$$f(x) = ax^2 + bx - 4 \quad \rightarrow \quad \begin{cases} 6 = a(2)^2 + b(2) - 4 \\ -10 = a(-6)^2 + b(-6) - 4 \end{cases}$$

Exploration

- Now, when you get a system that starts like this one, I'd recommend "cleaning it up" first so that that variables line up and that the constants are on the other side of each equation. Then you can solve the system using either substitution or elimination.

$$\begin{cases} 6 = a(2)^2 + b(2) - 4 \\ -10 = a(-6)^2 + b(-6) - 4 \end{cases} \rightarrow \begin{cases} 4a + 2b = 10 \\ 36a - 6b = -6 \end{cases}$$

Exploration

- And once you solve for a and b, you plug them back in to get the equation for $f(x)$.

$$f(x) = \frac{1}{2}x^2 + 4x - 4$$

Practice

- The quadratic function $w(x)$ passes through the points $(0, 7), (-3, -2)$, and $(2, -17)$. Find an equation for $w(x)$.

Converting from Standard Form to Vertex Form

- The process for going from vertex form to standard form is not too bad because it's just distributing and combining like terms. But the process for going from standard form to vertex form is a little more complicated. So let's go over a process for doing this.

$$f(x) = x^2 - 8x + 19$$

Completing the Square

- One way to do this is through a method called **COMPLETING THE SQUARE**. While there are other methods to help put a quadratic function into vertex form, completing the square doesn’t involve memorizing formulas but rather using an algebraic process to make the change.

$$f(x) = x^2 - 8x + 19$$

Completing the Square

- Completing the square involves finding a number for the constant of a quadratic expression that would make the expression have two of the same factors. Having two of the same factors allows us to write something like $(x + \cdots)(x + \cdots)$ as $(x + \cdots)^2$. So in the following quadratic expression, what number could we put in the blank that would make the expression factor to $(x + 5)^2$?

$$x^2 + 10x + \_\_\_$$

Completing the Square

- If we filled in the blank with the number “25,” then the quadratic expression would factor to $(x+5)^2$.

$$x^2 + 10x + \underline{25} \quad \rightarrow \quad (x+5)(x+5) \quad \rightarrow \quad (x+5)^2$$

Completing the Square

- Now, what is the relationship between the 10 and the 25 in the quadratic expression below that makes the expression have the same two factors?

$$x^2 + 10x + 25$$

Completing the Square

- To get the correct number to fill in the blank for completing the square, all you do is square half the b-term (in this case $\frac{10}{2} = 5$ and $5^2 = 25$). The number that you get to complete the square will always be positive because squaring a real number will result in a positive number.

$$x^2 + 10x + 25$$

Practice

- Fill in the blank below with a number that will make the quadratic expression have two of the same factor. Then rewrite the expression in the form $(x - \cdots)^2$.

$$x^2 - 14x + \_\_\_\_$$

Completing the Square

- Now, let's go back to a quadratic function in standard form and talk about the steps that we need to take to convert it to vertex form.

$$f(x) = x^2 - 8x + 19$$

Completing the Square

- First, temporarily ignore the original c-value and replace it with two blanks.

$$f(x) = x^2 - 8x + 19 \quad \rightarrow \quad f(x) = x^2 - 8x + \_\_\_ + \_\_\_$$

Completing the Square

- In the first blank, write the number that completes the square with the first two terms.

$$f(x) = x^2 - 8x + \underline{16} + \underline{\quad\quad}$$

Completing the Square

- In the second blank, write the number that you need to add to or subtract from the number in the other blank to get back to the original c-value.

$$f(x) = x^2 - 8x + 19 \quad \rightarrow \quad f(x) = x^2 - 8x + \underline{16} + \underline{3}$$

Completing the Square

- Next, group the first three terms of your altered function together inside a set of parentheses.

$$f(x) = (x^2 - 8x + 16) + 3$$

Completing the Square

- And last, factor the trinomial that you just grouped and write it as a squared binomial.

$$f(x) = \left(x^2 - 8x + 16\right) + 3 \quad \rightarrow \quad f(x) = (x-4)^2 + 3$$

Completing the Square

- And once the quadratic function is converted to vertex form, you can pick identify the vertex.

$$f(x) = (x-4)^2 + 3 \quad \rightarrow \quad \text{Vertex: } (4, 3)$$

Practice

- Convert the following quadratic function to vertex form. Then identify the vertex of the function.

$$g(x) = x^2 - 10x + 31$$

Practice

- Convert the following quadratic function to vertex form. Then identify the vertex of the function.

$$p(x) = x^2 + 4x - 9$$

Completing the Square

- Converting a quadratic function to vertex form is pretty straightforward when the a-value equals 1, but it gets a little more complicated when the a-value does not equal 1. So let's go through the process of converting a quadratic function like the one below to vertex form.

$$f(x) = 3x^2 - 18x + 33$$

Completing the Square

- First, we can factor out the a-value from all three terms of the function. And while quadratics exist where not all terms are divisible by the a-value, I'll only give you ones that are for completing the square purposes.

$$f(x) = 3x^2 - 18x + 33 \quad \rightarrow \quad f(x) = 3(x^2 - 6x + 11)$$

Completing the Square

- Next, replace the c-value inside the parentheses with two blanks.

$$f(x) = 3\left(x^2 - 6x + 11\right) \quad \rightarrow \quad f(x) = 3(x^2 - 6x + \_\_\_ + \_\_\_)$$

Completing the Square

- Next, complete the square inside the parentheses and add or subtract a number in the second blank to get back to the c-value in the parentheses.

$$f(x) = 3\left(x^2 - 6x + 11\right) \rightarrow f(x) = 3(x^2 - 6x + \underline{\ 9\ } + \underline{\ 2\ })$$

Completing the Square

- Next, add a second set of parentheses around the first three terms inside the parentheses that you already have.

$$f(x) = 3\big((x^2 - 6x + 9) + 2\big)$$

Completing the Square

- Next, factor the trinomial inside the inner set of parentheses to a squared binomial.

$$f(x) = 3\left(\left(x^2 - 6x + 9\right) + 2\right) \rightarrow f(x) = 3\left((x-3)^2 + 2\right)$$

Completing the Square

- And last, the a-value that you factored out originally needs to be distributed back to what's inside the outer set of parentheses. When you do this distribution, that a-value becomes the coefficient of the squared binomial (so it looks like $a(x \pm \cdots)^2$) and it also is multiplied by the constant (which many students forget to to).

$$f(x) = 3\left((x-3)^2 + 2\right) \quad \rightarrow \quad f(x) = 3(x-3)^2 + 6$$

Practice

- Convert the following quadratic function to vertex form. Then identify the vertex of the function.

$$g(x) = -2x^2 + 28x + 16$$

Practice

- Convert the following quadratic function to vertex form. Then identify the vertex of the function.

$$h(x) = \frac{1}{3}x^2 - 4x - 1$$

Factored Form

- The last form of a quadratic function that we briefly mentioned earlier (called **FACTORED FORM**) is written as $f(x) = a(x - r)(x - s)$. How do you think we could convert the following quadratic function to factored form?

$$f(x) = \frac{1}{4}x^2 - x - 8$$

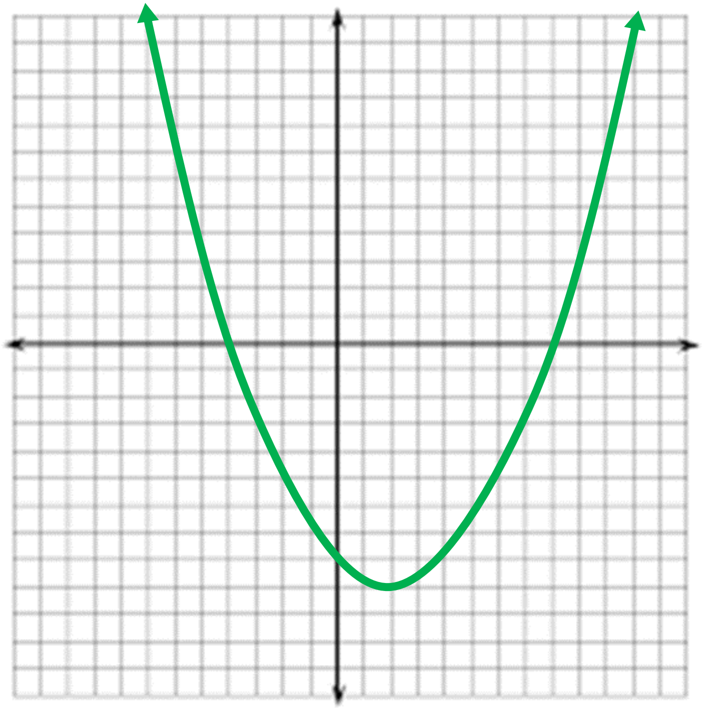
Factored Form

- To convert a quadratic function from standard form to factored form, we simply **FACTOR** the quadratic. In this case, we could first factor the a-value out of all three terms and then we could factor the rest of the quadratic.

$$f(x) = \frac{1}{4}x^2 - x - 8 \quad \rightarrow \quad f(x) = \frac{1}{4}(x+4)(x-8)$$

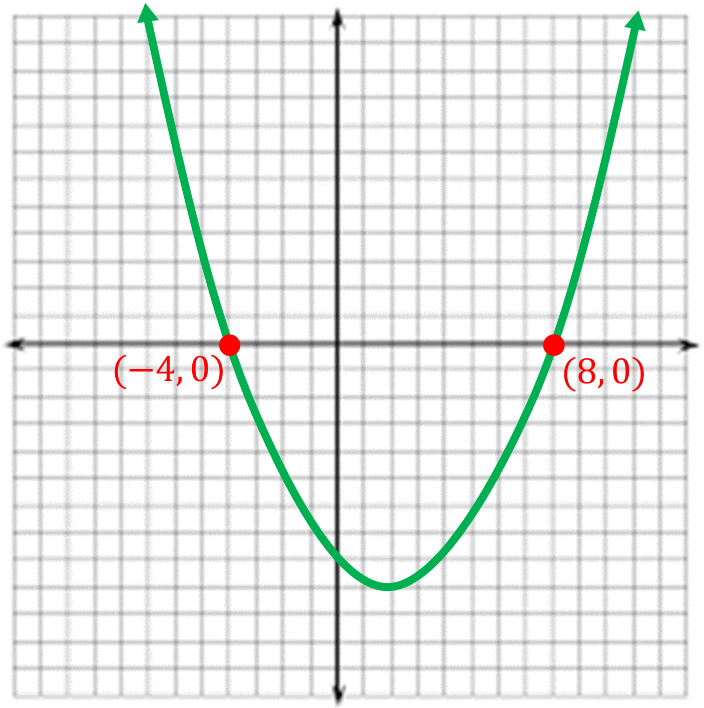
Factored Form

- Now, if we graphed the function $f(x) = \frac{1}{4}(x + 4)(x - 8)$ we'd get the parabola below. What do you think our quadratic function written in factored form tells us about the parabola?



Factored Form

- If we set each factor of $f(x) = \frac{1}{4}(x+4)(x-8)$ equal to 0 (in this case $x + 4 = 0$ and $x - 8 = 0$) and solve, we get the **X-INTERCEPTS** of the parabola.



Factored Form

- Thus, the r- and s-values of a quadratic function written in factored form represent the x-intercepts of the quadratic function.

$$f(x) = a(x - r)(x - s) \quad \rightarrow \quad \text{X-intercepts: } (r, 0) \text{ and } (s, 0)$$

Important Note

- The x-intercepts of a quadratic function are also referred to as the **ZEROES** and the **ROOTS** of the function, so be on the lookout for that vocabulary.

Practice

- Convert the following quadratic function to factored form. Then identify the zeroes of the function.

$$f(x) = 2x^2 - 2x - 40$$

Factored Form

- Sometimes we won't be able to factor the a-value out of all three terms and so the process of converting the quadratic function to factored form is slightly more challenging. Let's try this with the quadratic function below.

$$f(x) = 6x^2 - x - 12$$

Factored Form

- If there is not GCF, then start by factoring the quadratic like we did in our first unit.

$$f(x) = 6x^2 - x - 12 \quad \rightarrow \quad f(x) = (2x - 3)(3x + 4)$$

Factored Form

- Now, recall that the a-value of standard form was 6. That means that the a-value for factored form should also be 6. How can we take our factored quadratic function and rewrite it so that it's in the form $f(x) = a(x - r)(x - s)$?

$$f(x) = (2x - 3)(3x + 4) \quad \rightarrow \quad f(x) = 6(x - \cdots)(x + \cdots)$$

Factored Form

- First, you can factor out the coefficient of the x-term from each set of parentheses. The way to factor out the coefficient from the other number inside each set of parentheses is to divide the number by the coefficient.

$$f(x) = (2x - 3)(3x + 4) \quad \rightarrow \quad f(x) = 2\left(x - \frac{3}{2}\right)3\left(x + \frac{4}{3}\right)$$

Factored Form

- Once you factor out each coefficient, you can then multiply them together and put the number in the from of the two sets of parentheses.

$$f(x) = 2\left(x - \frac{3}{2}\right)3\left(x + \frac{4}{3}\right) \quad \rightarrow \quad f(x) = 6\left(x - \frac{3}{2}\right)\left(x + \frac{4}{3}\right)$$

Practice

- The following two quadratic functions are equivalent. Find the value of r and of s.

$$w(x) = 4x^2 + 11x - 3 \quad \leftrightarrow \quad w(x) = 4(x - r)(x - s)$$

Factored Form

- If you're asked to find an equation for a quadratic function and you know the x-intercepts of the function, you can plug in the x-intercepts for r and s and then use another point on the graph of the function to plug in and solve for the a-value.

X–intercepts: $(5, 0)$ and $(-2,0)$ $\rightarrow$ $f(x) = a(x-5)(x+2)$

Practice

- A parabola passes through the points $(-2, 0), (9, 0)$, and $(2, 14)$. Find an equation for the quadratic function.

Practice

- The quadratic function $h(x)$ passes through the points $(-3, 0)$ and $(8, 11)$. If the axis of symmetry of the function is at $x = 1$, find $h(2)$.

Factored Form

- To convert a quadratic function from standard form to factored form, we simply factored the quadratic. How do you think we could convert a quadratic function from factored form to standard form?

$$f(x) = 2(x - 3)(x + 1)$$

Factored Form

- To convert a quadratic function from factored form to standard form, we could simply distribute the factors out and then distribute the a-value to all three terms.

$$f(x) = 2(x-3)(x+1) \quad \rightarrow \quad f(x) = 2\left(x^2 - 2x - 3\right) \quad \rightarrow \quad f(x) = 2x^2 - 4x - 6$$

Practice

- The following two quadratic functions are equivalent. Find the value of b and of c.

$$g(x) = \frac{-4}{3}(x-6)(x+12) \quad \leftrightarrow \quad g(x) = \frac{-4}{3}x^2 + bx + c$$

Converting Between Forms

- There will be times in which a quadratic function needs to switch from vertex form to factored form or vice versa. For those times, it's easiest to just convert the original form to standard form first and then convert standard form to the form that you need to switch to.

$$p(x) = 5(x+7)(x+1) \quad \leftrightarrow \quad p(x) = 5x^2 + 40x + 35 \quad \leftrightarrow \quad p(x) = 5(x+4)^2 - 45$$

Practice

- The following two quadratic functions are equivalent. Find the value of r and of s.

$$f(x) = -6(x+5)^2 + 54 \quad \leftrightarrow \quad f(x) = -6(x-r)(x-s)$$

Practice

- The following two quadratic functions are equivalent. Find the value of h and of k.

$$z(x) = \frac{1}{2}(x-2)(x-10) \quad \leftrightarrow \quad z(x) = \frac{1}{2}(x-h)^2 + k$$

Exploration

- Some quadratic functions cannot be factored. How, then, could we find the x-intercepts of a quadratic function that is un-factorable?

$$f(x) = 2x^2 + 3x - 6$$

Quadratic Formula

- When you need to find the x-intercepts of a quadratic function that cannot be factored, you can use the **QUADRATIC FORMULA**. You do not need to memorize the formula but you need to know that the a-value is the coefficient of the x^2 term, the b-value is the coefficient of the x term, and the c-value is the constant.

$$f(x) = 2x^2 + 3x - 6 \quad \rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Quadratic Formula

- Since the answers you get from the quadratic formula are the x-intercepts of the function, you can substitute them for the r and s values from factored form while keeping the a-value the same.

$$x = -1.137, 2.637 \quad \rightarrow \quad f(x) = 2(x + 1.137)(x - 2.637)$$

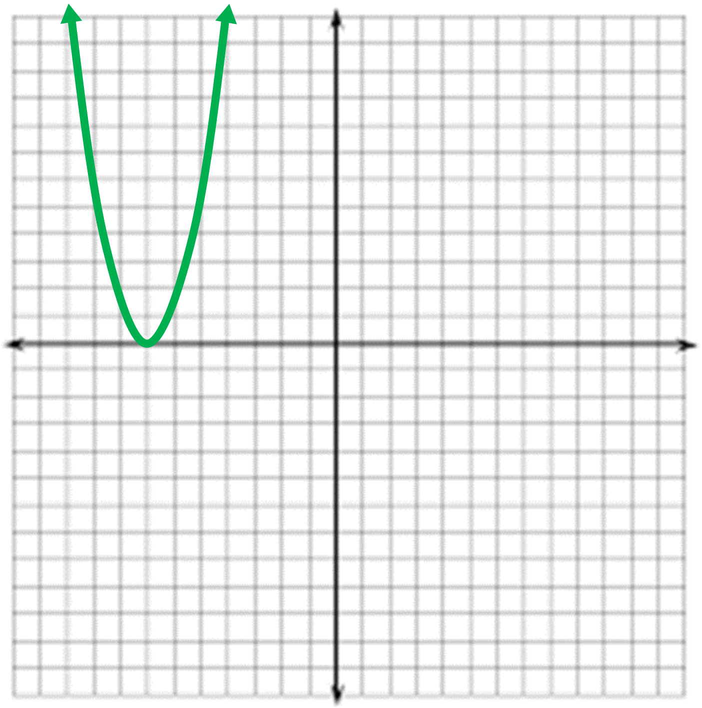
Practice

- Use the quadratic formula ($x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$) to find the x-intercepts of the quadratic function below. Then write the function in factored form.

$$v(x) = -3x^2 - 4x + 10$$

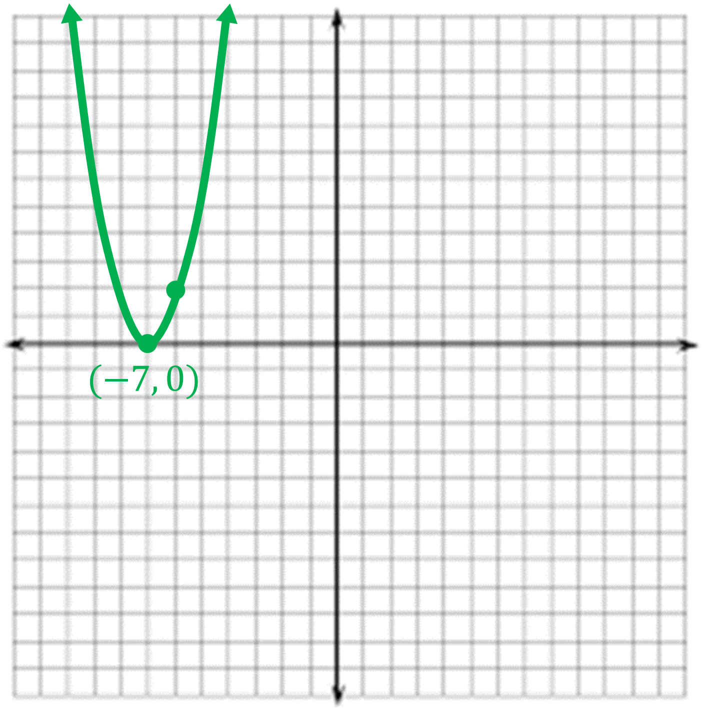
Exploration

- What would factored form look like for a parabola whose vertex is on the x-axis?



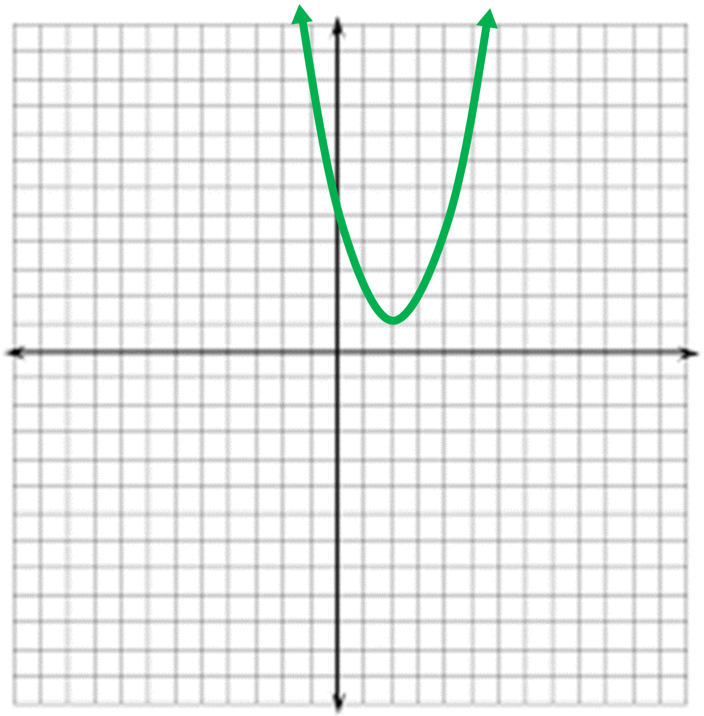
Exploration

- When a parabola has a vertex on the x-axis, the r and s values from factored form are the same, which essentially makes factored form and vertex form the same. For graph of the function $f(x)$ below, we could say $f(x) = 2(x+7)(x+7)$ or $f(x) = 2(x+7)^2$.



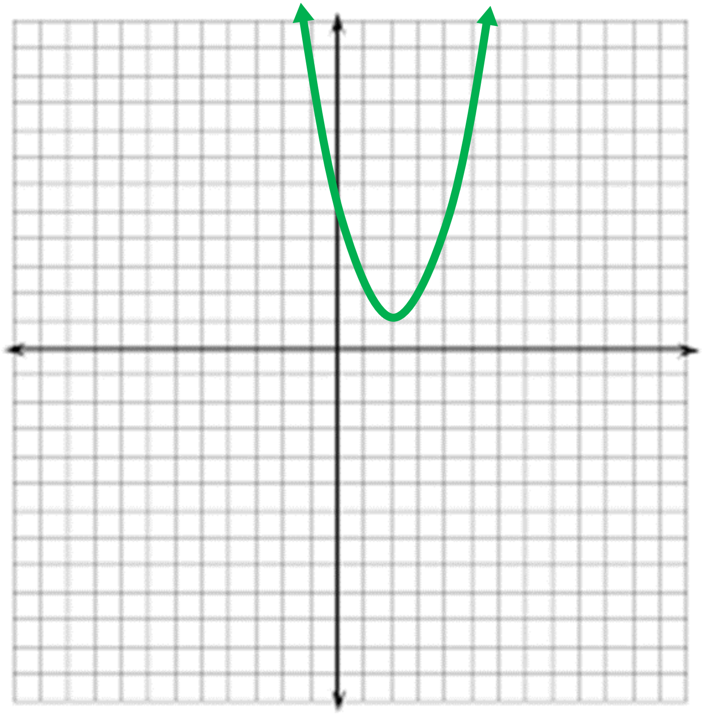
Exploration

- What would factored form look like for a parabola that has no x-intercepts?



Exploration

- A parabola that has no x-intercepts **CANNOT** be written in factored form. That is unless…



Exploration

- Try using the quadratic formula ($x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$) to find the x-intercepts of the following quadratic function. What happens?

$$f(x) = x^2 - 4x + 13$$

Exploration

- When using the quadratic formula to find the x-intercepts of $f(x)$, we get to a point where you have to take the square root of a negative number, which we usually aren't able to do.

$$f(x) = x^2 - 4x + 13 \quad \rightarrow \quad x = \frac{4 \pm \sqrt{-36}}{2}$$

Real Numbers

- When solving quadratic equations in Algebra I, your teacher may have told you to stop when you got to a point where you had to take the square root of a negative number. And when you stopped your teacher may have told you to say the quadratic equation had no **REAL** solutions. What this means is that the solutions are to the equation are not **REAL** numbers.

$$x = \frac{4 \pm \sqrt{-36}}{2}$$

Imaginary Numbers

- Luckily, there is such a thing as an **IMAGINARY** number, which we refer to as "i". An imaginary number is equal to the square root of -1.

$$i = \sqrt{-1}$$

Imaginary Numbers

- So when you're using the quadratic formula and you get to a point where you have to take the square root of a negative number, you can split up the radical and change $\sqrt{-1}$ to i.

$$x = \frac{4 \pm \sqrt{-36}}{2} \quad \rightarrow \quad x = \frac{4 \pm \sqrt{36} \cdot \sqrt{-1}}{2} \quad \rightarrow \quad x = \frac{4 \pm 6i}{2}$$

Complex Numbers

- And when you simplify your answer down further, you'll end up two numbers that can be written in the form $a + bi$. Any number written in the form $a + bi$ is known as a **COMPLEX NUMBER**. The "a" part of the complex number is the **REAL** part and the "bi" part is the **IMAGINARY** part.

$$x = \frac{4 \pm 6i}{2} \quad \rightarrow \quad x = 2 + 3i, 2 - 3i$$

Practice

- The following function can be written in the form $q(x) = 2(x - r)(x - s)$, where both r and s are complex numbers. Find the value of r and of s.

$$q(x) = 2x^2 - 32x + 178$$

Exploration

- Say the quadratic function $f(x) = 3x^2 + bx + c$ had imaginary roots at $x = 4 + i$ and $x = 4 - i$. How might we find the value of b and of c?

Exploration

- Since $x = 4 + i$ and $x = 4 - i$ are the roots of the function, we can substitute them in for r and s in the factored form of a quadratic function along with $a = 3$.

$$f(x) = 3(x - (4 + i))(x - (4 - i))$$

Exploration

- Next, we can multiply $(x - 4 - i)$ by $(x - 4 - i)$. This can be a tedious process, but it'll pay off in the end.

$$f(x) = 3(x - 4 - i)(x - 4 + i) \quad \rightarrow \quad f(x) = 3(x^2 - 4x - ix - 4x + 16 + 4i + ix - 4i - i^2)$$

Exploration

- Next, we can combine like terms and then distribute the 3 to everything that remains.

$$f(x) = 3\left(x^2 - 4x - \cancel{ix} - 4x + 16 + \cancel{4i} + \cancel{ix} - \cancel{4i} - i^2\right) \quad \rightarrow \quad f(x) = 3x^2 - 24x + 48 - 3i^2$$

Exploration

- Now let's consider what to do from here. If $i = \sqrt{-1}$, what do you think i^2 equals?

$$f(x) = 3x^2 - 24x + 48 - 3i^2$$

Exploration

- Since $i = \sqrt{-1}$, that means $i^2 = -1$. Therefore, we can substitute -1 in for i^2.

$$f(x) = 3x^2 - 24x + 48 - 3i^2 \quad \rightarrow \quad f(x) = 3x^2 - 24x + 48 - 3(-1)$$

Exploration

- And last, we simply combine like terms to get the quadratic function $f(x) = 3x^2 - 24x + 51$.

$$f(x) = 3x^2 - 24x + 48 - 3(-1) \quad \rightarrow \quad f(x) = 3x^2 - 24x + 51$$

Practice

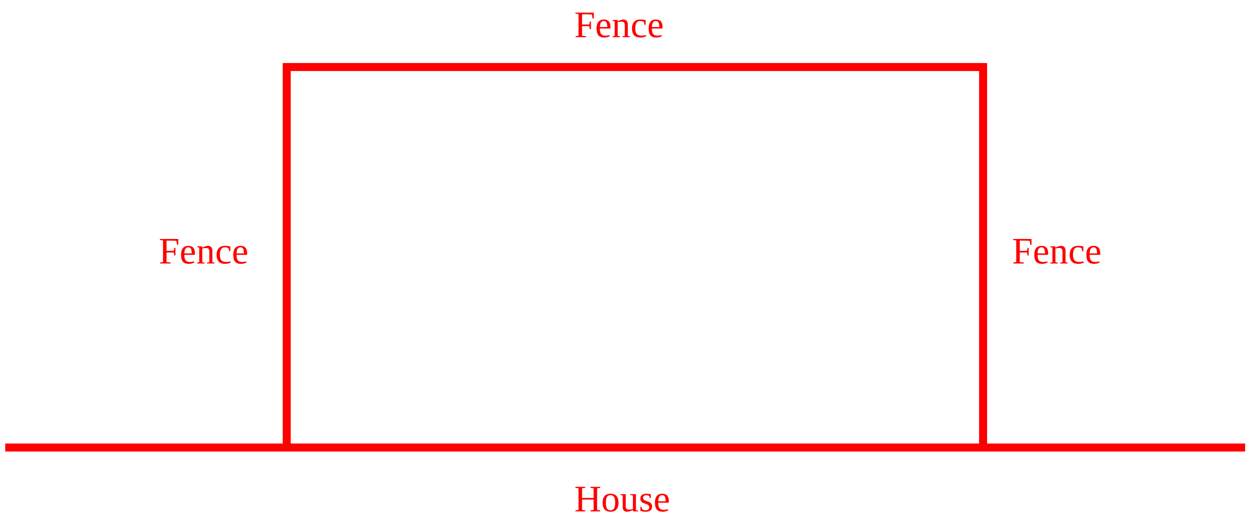
- The quadratic function $h(x) = -x^2 + bx + c$ has complex roots at $x = -6 + 5i$ and $x = -6 - 5i$. Find the value of b and of c.

Applications of Quadratic Functions

- Quadratic functions can be used to model a lot of real-world scenarios, from launching objects into the air to figuring out how many products a company needs to sell in order to its maximize profit. Let’s look at an example to see how to use quadratic functions to model these situations.

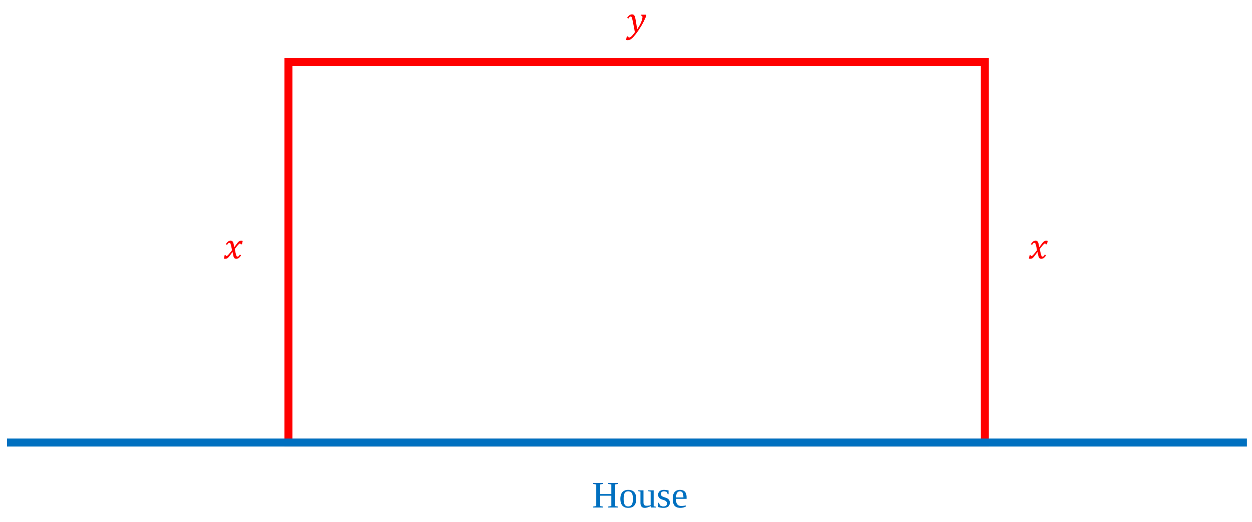
Applications of Quadratic Functions

- Suppose you want to build a rectangular fence in your yard with one side of the fence being your house. If you have 120 yds of material to build the fence, what dimensions should you make the fence in order to **MAXIMIZE** the area of the part of the yard contained within the fence?



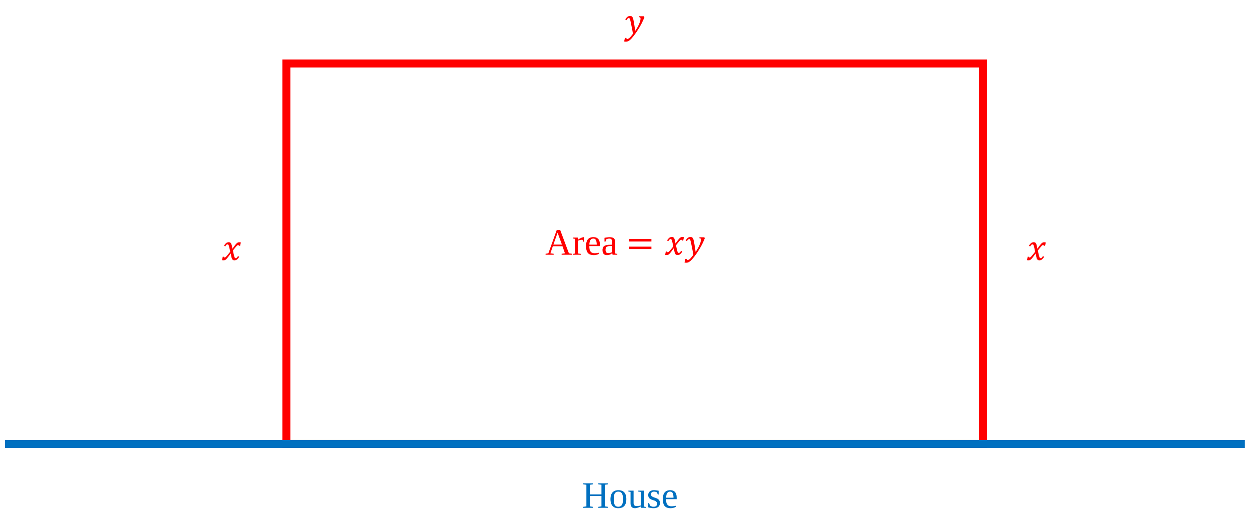
Applications of Quadratic Functions

- So let's set this up where we call one dimension of the fence "x" and the other dimension "y." If our goal is to maximize the area of the part of the yard enclosed by the fence, what is an equation we could write down for this area?



Applications of Quadratic Functions

- And if we have 120 yds. of fencing, what is an equation we write down that represents all the fencing that we need to build the fence (assuming we'll use all the fencing we have)?



Applications of Quadratic Functions

- So if the area of the fenced-in yard can be found with the equation $\text{Area} = xy$ and if the amount of fencing we have to build the fence can be expressed by the equation $2x + y = 120$, how might we use these equations to find the dimensions we need the fence to be in order to maximize the area of the part of the yard contained within the fence?

Applications of Quadratic Functions

- First, we can solve the fencing equation for y.

$$2x + y = 120 \quad \rightarrow \quad y = -2x + 120$$

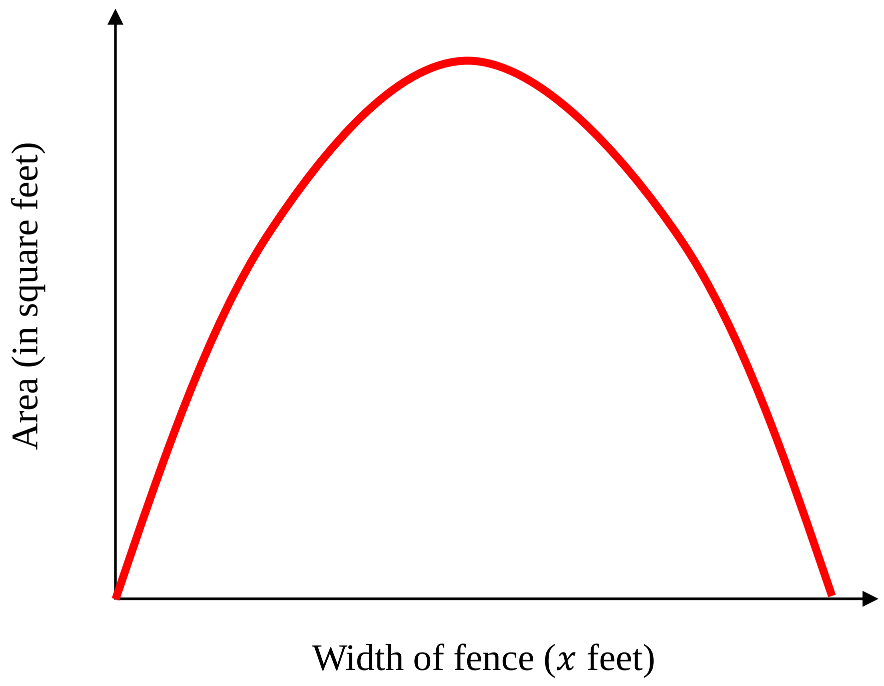
Applications of Quadratic Functions

- Next, we can substitute this $-2x + 120$ in for y in the area equation. When we do this, what type of equation do we get?

$$y = -2x + 120, \text{Area} = xy \quad \rightarrow \quad \text{Area} = x(-2x + 120)$$

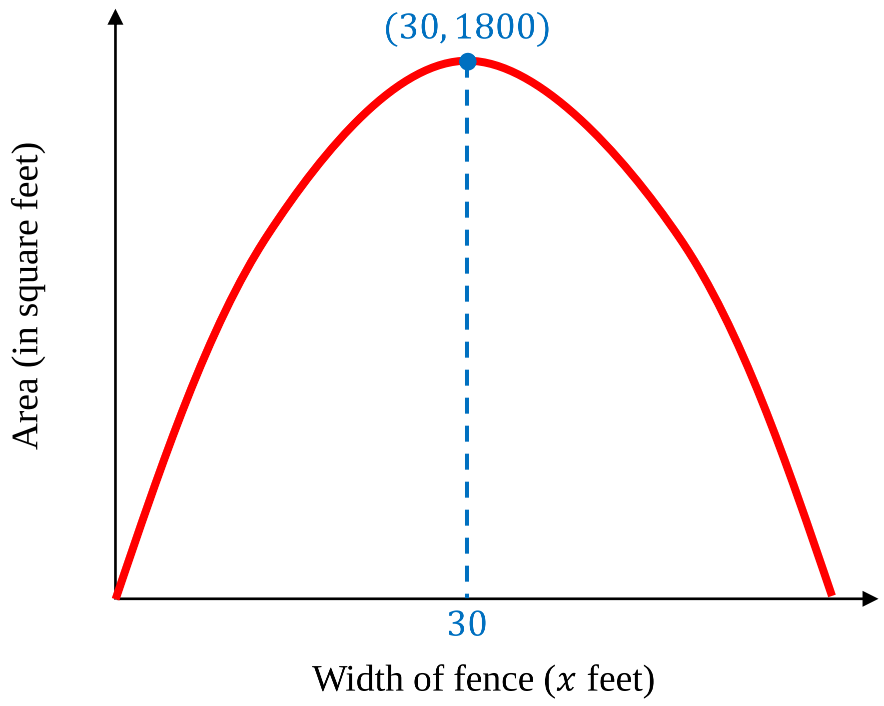
Applications of Quadratic Functions

- Next, we can graph the Area $= -2x^2 + 120x$ function on our calculators. If our goal is to find the dimensions of the fence that would **MAXIMIZE** the area of the part of the yard contained within the fence, which part of the parabola do we care about?



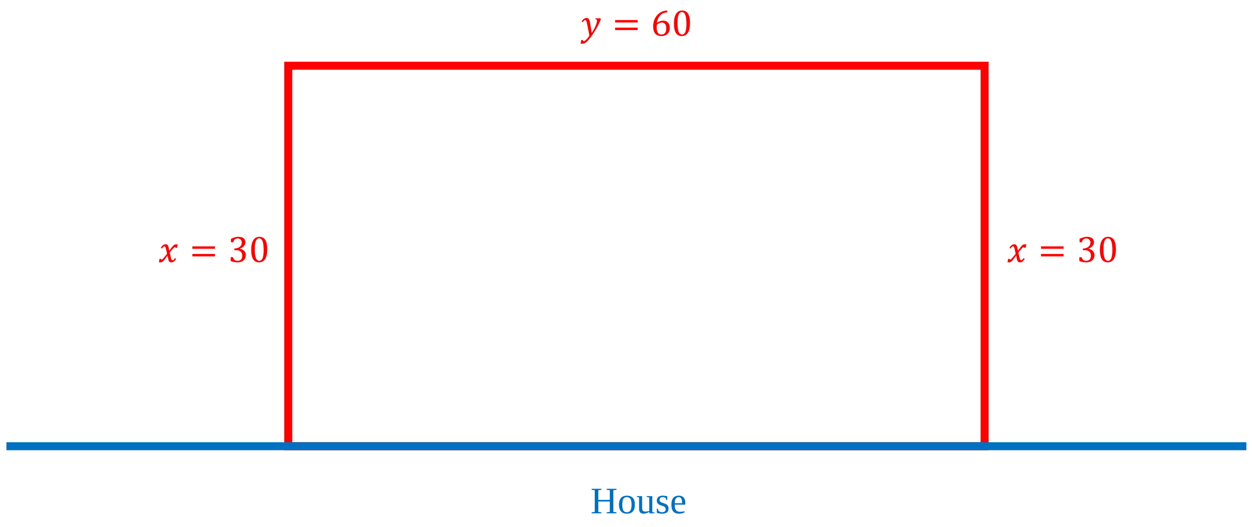
Applications of Quadratic Functions

- If our goal is to find the dimensions of the fence that would **MAXIMIZE** the area of the part of the yard contained within the fence, we would look for the x-coordinate of the vertex (while the y-coordinate of the vertex would be the maximum area we could contain within the fence).



Applications of Quadratic Functions

- And once we know the value of x (which is one dimension of the fence that we want to build), we could use the $2x + y = 120$ equation to find the other dimension of the fence (in this case y).



Practice

- Upon arriving at the Grand Canyon, you decide to throw a rock to see how long it takes for it to reach the canyon's floor. The distance of the rock to the bottom of the canyon (in meters) x seconds after you throw is can be modelled by the function $D(x) = -4.9x^2 + 14x + 2440$. Use your calculator to determine how long it takes for the rock to hit the canyon's floor.

Practice

- Because of supply and demand, the more you charge for a product the less people that are willing to buy the product. This creates the need for a company to set a competitive price for a product in order to maximize their revenue (which is the amount a company makes before accounting for costs). Suppose you own a tire company. The number of tires you sell each month (x) is dependent on the price of the tires (p) and this relationship can be represented by $x = -\frac{8}{9}p + 300$. The revenue of your company each month (R) can be found by multiplying the price of the tires (p) by the number of tires sold each month (x, so $R = xp$). At what price should your company set the tires at in order to **MAXIMIZE** your company's revenue?